Holomorphic Vector Bundles on Ruled Surfaces and Their Blowing Ups

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Abstract

This work is motivated by the general interest in moduli spaces of based SU(r)-instantons on S^4 and on the connected sum of complex projective planes, which can be interpreted as moduli spaces of framed holomorphic vector bundles on blown up ruled surfaces. To be precise, we consider moduli of based SU(r)-instantons over all self-dual four dimensional manifolds, where the twistor fibration contains a surface of degree 1 which itself contains a twistor fibre. This applies for instance to the well-examined class of LeBrun-twistor spaces (cf. [30], [27]).

We display the necessary background of this relationship. Inspired by J. Hurtubise's paper [18], we study the local jumping behaviour of such framed vector bundles and introduce the concept of framed exceptional local jumps. We describe framed ordinary and exceptional jumps by monads and examine the geometric properties of their fine moduli spaces.

Based on this examination and on the results of [4] and [40], where the Atiyah-Jones conjecture for based SU(r)-instantons on S^4 is proved, we show homological and homotopical charge stability for all considered moduli of based SU(r)-instantons. Moreover, we present a smooth compactification of these moduli spaces.

Zusammenfassung

Diese Arbeit ist motiviert durch das allgemeine Interesse an den Modulräumen von basierten SU(r)-Instantonen auf S^4 und auf der zusammenhängenden Summe von komplexen projektiven Ebenen, welche als Modulräume von gerahmten holomorphen Vektorbündeln auf Aufblasungen von Regelflächen interpretiert werden können. Genauer gesagt, betrachten wir Modulräume von basierten SU(r)-Instantonen auf all denjenigen selbstdualen vierdimensionalen Mannigfaltigkeiten, welche eine Fläche vom Grad 1 enthalten, die selbst wiederum eine Twistorfaser enthält. Dies trifft zum Beispiel für die wohluntersuchte Klasse von LeBrun-Twistorräumen zu (vgl. [30],[27]).

Es erfolgt eine Darstellung des notwendigen Hintergrundes dieses Zusammenhanges. Inspiriert durch J. Hurtubise's Artikel [18] untersuchen wir das lokale Sprungverhalten dieser gerahmten Vektorbündel und führen den Begriff des gerahmten exzeptionellen lokalen Sprunges ein. Wir beschreiben die gerahmten gewöhnlichen und exzeptionellen Sprünge durch Monaden und untersuchen die geometrischen Eigenschaften ihrer feinen Modulräume.

Aufbauend auf dieser Untersuchung und den Resultaten in [4] und [40], worin die Atiyah-Jones-Vermutung für basierte SU(r)-Instantonen auf S^4 bewiesen wird, zeigen wir homologische und homotopische Charge-Stabilität für alle betrachteten Modulräume basierter SU(r)-Instantonen. Darüberhinaus stellen wir eine glatte Kompaktifizierung dieser Modulräume vor.

Contents

0 Motivation and introduction

This thesis is motivated by the abundant advancements in the theory of twistor spaces and instanton bundles in the last years. In 1984, S. Donaldson has shown in [8] that based $\mathrm{SU(r)}$-instantons on the four dimensional sphere S^4 can be interpreted as holomorphic vector bundles on the complex projective plane equipped with a trivialisation along a fixed line. Moreover, he has given a monadic description of these framed vector bundles based on Hulek monads. In his article [18] from 1986, J. Hurtubise used Donaldson's approach to identify the moduli of based $\mathrm{SU(2)}$-instantons on S^4 with the moduli of framed vector bundles on rational ruled surfaces, where the framing lives on the section of the ruling and where the vector bundles considered are trivial along a fixed special fibre. He studied these moduli spaces by examining the local jumping behaviour in neighbourhoods of fibres and introduced and described the concept of framed local jumps. Besides other benefits, his results turned out to be essential in proving the Atiyah-Jones conjecture about the homotopical stability of based $\mathrm{SU(r)}$-instantons in [4]. The article of Hurtubise can be called the main inspiration of this thesis.

The idea of Donaldson has been generalised by N. Buchdahl in [6] and [7] for $\mathrm{SU(r)}$-instantons on the connected sum of complex projective planes. In fact, this idea works for all based four dimensional self-dual manifolds (M, x) where the twistor space contains a surface S' of degree one, which in turn contains the twistor fibre F' over x. These twistor spaces have been examined by B. Kreußler, H. Kurke ([26], [27]) and C. LeBrun ([30]). In particular, the surface S' is always a blowing up of the complex projective plane. A based $\mathrm{SU(r)}$-instanton of charge m on (M, x) is associated to a framed $\mathrm{SU(r)}$-instanton bundle on the twistor space by using the Penrose-Ward transformation, and this framed holomorphic vector bundle restricts to a framed vector bundle on $(\mathrm{S}', \mathrm{F}')$ with Chern classes $\mathrm{c}_1 = 0$ and $\mathrm{c}_2 = \mathrm{m}$. This map is bijective, as demonstrated by Buchdahl.

The theory of framed vector bundles and their moduli spaces has been developed by M. Lübke ([32]), D. Huybrechts and M. Lehn ([31], [21], [20]). One result is, that there is a fine algebraic moduli space $\mathcal{V}ec(\mathrm{S}', \mathrm{F}', \mathcal{O}^{\mathrm{r}}_{\mathrm{F}'}, \mathrm{c}_1 = 0, \mathrm{c}_2 = \mathrm{m})$ for the framed vector bundles on $(\mathrm{S}', \mathrm{F}')$ with Chern classes $\mathrm{c}_1 = 0$ and $\mathrm{c}_2 = \mathrm{m}$. In [33], a suitable deformation theory of framed vector bundles is given and it is shown that the map from the space $\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}, \mathrm{r})$ of based SU(r)-instantons of charge m on (M, x) to $\mathcal{V}ec(\mathrm{S}', \mathrm{F}', \mathcal{O}^{\mathrm{r}}_{\mathrm{F}'}, \mathrm{c}_1 = 0, \mathrm{c}_2 = \mathrm{m})$, due to Buchdahl, is in fact a real-analytic isomorphism.

The initial idea of this thesis is to approach this moduli space in the same way as Hurtubise has done in [18]. That is, we consider a blowing up $\mathrm{S} \to \mathrm{S}'$ in a point on F' with exceptional divisor N and identify $\mathcal{V}ec(\mathrm{S}', \mathrm{F}', \mathcal{O}^{\mathrm{r}}_{\mathrm{F}'}, \mathrm{c}_1 = 0, \mathrm{c}_2 = \mathrm{m})$ with $\mathcal{V}(\mathrm{m}, \mathrm{r})$, wherewith we denote the moduli space of vector bundles on S equipped with a trivialisation along N, which are trivial along the strict transform F of F' and which have Chern classes $\mathrm{c}_1 = 0$ and $\mathrm{c}_2 = \mathrm{m}$. We have a natural morphism $\pi : \mathrm{S} \to \mathrm{N}$ and want to describe $\mathcal{V}(\mathrm{m}, \mathrm{r})$ by the local jumping behaviour around fibres of π. The difference from [18] is that π is not a ruled surface, but a blowing up of a ruled surface. Therefore, we have to struggle with exceptional fibres. In this form, the problem was given to me by my mentor and supervisor, Herbert Kurke from the Humboldt-University in Berlin.

The first chapter of this thesis reviews some necessary background, including the theory of framed module sheaves, their moduli and their deformation theory and the above mentioned relationship between based SU(r)-instantons and framed holomorphic vector bundles on rational surfaces. We proceed by presenting the space $\mathcal{V}(\mathrm{m}, \mathrm{r})$ and a morphism $\Delta : \mathcal{V}(\mathrm{m}, \mathrm{r}) \to \mathrm{Sym}^{\mathrm{m}}\mathrm{N}$, which is constructed with the help of determinantal divisors. The study of Δ may be considered as the central issue of this work. In the case where $\pi : \mathrm{S} \to \mathrm{N}$ is ruled, which corresponds to $\mathrm{M} = \mathrm{S}^4$, the fibres of Δ can be described using $\mathcal{J}(\mathrm{m}, \mathrm{r})$, the spaces of framed local jumps of order m from [18], which we will call ordinary framed jumps. Since S is in general the blowing up of a ruled surface, we have to consider in addition spaces of framed exceptional jumps of order m, which we denote with $\mathcal{E}(\mathrm{m}, \mathrm{r})$.

Chapter 2 is dedicated to the study of the spaces $\mathcal{J}(\mathrm{m}, \mathrm{r})$ and $\mathcal{E}(\mathrm{m}, \mathrm{r})$. We review known facts about $\mathcal{J}(\mathrm{m},\mathrm{r})$ - that it is a variety of dimension $2\mathrm{mr}-\mathrm{m}$ appearing as geometric quotient, which is due to a monadic description of ordinary framed jumps. Then we treat $\mathcal{E}(\mathrm{m},\mathrm{r})$ in an analogous manner: we define the order of an exceptional jump, give a monadic description based on King monads ([24]) and describe $\mathcal{E}(\mathrm{m},\mathrm{r})$ as a geometric quotient. In particular, we show that $\mathcal{E}(\mathrm{m},\mathrm{r})$ is a reduced algebraic scheme of dimension $2\mathrm{mr}-\mathrm{m}$, consisting of two irreducible components which are closures of two disjoint open embeddings of $\mathcal{J}(\mathrm{m},\mathrm{r})$ into $\mathcal{E}(\mathrm{m},\mathrm{r})$. Moreover, we discuss the spaces $\mathcal{E}(1,\mathrm{r})$, show a theorem of finite determination, enabling us to generalise our considerations for blowing ups of non-rational ruled surfaces, and we describe the singular locus of $\mathcal{J}(\mathrm{m},\mathrm{r})$ and $\mathcal{E}(\mathrm{m},\mathrm{r})$.

Based on these results, we examine in the third chapter the problem of homotopical stability of the moduli spaces $\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r})$. In [4] and [40], C. Boyer, J. Hurtubise, B. Mann, R. Milgram and Y. Tian have proved the Atiyah-Jones conjecture. Precisely, they have shown that for $\mathrm{M}=\mathrm{S}^4$, the stabilisation map of C. Taubes ([39]) from $\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r})$ to $\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}+1,\mathrm{r})$ induces isomorphisms $\pi_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r}))\cong\pi_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}+1,\mathrm{r}))$ on the homotopy groups with $\mathrm{t}\leq\frac{\mathrm{m}}{2}-2$. We generalise the definition of the stabilisation map given in [4] for all considered manifolds M. Using the result for $\mathrm{M}=\mathrm{S}^4$, some insight in L-stratifications and the descriptions of $\mathcal{J}(\mathrm{m},\mathrm{r})$ and $\mathcal{E}(\mathrm{m},\mathrm{r})$, we show that it induces isomorphisms $\pi_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r}))\cong\pi_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}+1,\mathrm{r}))$ for $\mathrm{t}\leq\frac{\mathrm{m}}{2}-2$, too. Although we do not execute it explicitly, we note that our considerations may be also used to generalise the results of Hurtubise and Milgram in [19] to blown up ruled surfaces.

In the last chapter, we complete the moduli space $\mathcal{V}(\mathrm{m},\mathrm{r})$ by an open embedding into a quotient scheme. The morphism Δ extends to the completion. We show that this completion is smooth and describe the degeneration locus.

The notation and symbols in this thesis are the same as in the standard literature. Any additional symbols and definitions introduced are included in an index found at the end. Please note that the dual of an object $*$ is always denoted with $*^{\vee}$.

I would like to express my appreciation to my supervisor, Herbert Kurke, for the support and mentorship given to me over the last years. I am also grateful for the pleasant and productive atmosphere in the research group for algebraic geometry at the Humboldt-University. Especially, I would like to thank Georg Hein, Ines Quandt and Klaus Altmann for all the helpful discussions. Last but not least I thank the Graduiertenkolleg "Geometrie und Nichtlineare Analysis". Without its fellowship over the last two years, my graduate study would not have been possible.

1 Basic material and the initial problem

1.1 Framed vector sheaves

Let X be an analytic space, Y a closed subspace with associated ideal sheaf $\mathcal{J}_{\mathrm{Y}|\mathrm{X}}$ and W a fixed coherent and locally free sheaf of modules over Y, i.e. a vector sheaf on Y. A *framed vector sheaf* to the data $(\mathrm{X}, \mathrm{Y}, \mathrm{W})$ consists of a pair (V, α) with V a vector sheaf over X and $\alpha : \mathrm{V}|_{\mathrm{Y}} \cong \mathrm{W}$ a framing isomorphism. A *morphism* f between two framed vector sheaves (V, α) and (V', α') is a sheaf morphism $\mathrm{f} : \mathrm{V} \to \mathrm{V}'$ with $\alpha' \circ (\mathrm{f}|_{\mathrm{Y}}) = \alpha$. For an analytic space T, a *family* of framed vector sheaves for the given data $(\mathrm{X}, \mathrm{Y}, \mathrm{W})$ parametrized by T is a framed vector sheaf $(\tilde{\mathrm{V}}, \tilde{\alpha})$ for the data $(\mathrm{T} \times \mathrm{X}, \mathrm{T} \times \mathrm{Y}, \mathrm{p}^*\mathrm{W})$, where $\mathrm{p} : \mathrm{T} \times \mathrm{Y} \to \mathrm{Y}$ is the projection.

A *deformation* of a fixed framed vector sheaf (V, α) over a germ of an analytic space $(\mathrm{T}, \mathrm{t}_0)$ is represented by a triple $(\tilde{\mathrm{V}}, \tilde{\alpha}, \psi)$, where (V, α) is a family of framed vector bundles over T and ψ is an isomorphism from $\tilde{\mathrm{V}}|_{\{\mathrm{s}_0\}\times \mathrm{X}}$ to V such that the diagram

$$\begin{array}{ccc}
\tilde{\mathrm{V}}|_{\{\mathrm{s}_0\}\times\mathrm{Y}} & \xrightarrow[\psi]{\cong} & \mathrm{V}|_{\mathrm{Y}} \\
\cong\Big\downarrow\tilde{\alpha} & & \cong\Big\downarrow\alpha \\
\mathrm{W} & \xrightarrow{=} & \mathrm{W}
\end{array}$$

commutes. Here, the pointed space $(\mathrm{T}, \mathrm{t}_0)$ is any representative of our germ. If $\eta : (\mathrm{T}', \mathrm{t}'_0) \to (\mathrm{T}, \mathrm{t}_0)$ is a local analytic isomorphism, then

$$((\eta \times \mathrm{id}_{\mathrm{X}})^*\tilde{\mathrm{V}}, (\eta \times \mathrm{id}_{\mathrm{X}})^*\tilde{\alpha}, (\eta \times \mathrm{id}_{\mathrm{X}})^*\psi)$$

represents the same deformation.

Two deformations $(\tilde{\mathrm{V}}, \tilde{\alpha}, \psi)$ and $(\tilde{\mathrm{V}}', \tilde{\alpha}', \psi')$ of (V, α) over the same germ can be realized as families over the same pointed analytic space $(\mathrm{T}, \mathrm{t}_0)$. A morphism between two deformations $\mathrm{f}\colon (\tilde{\mathrm{V}}, \tilde{\alpha}, \psi) \to (\tilde{\mathrm{V}}', \tilde{\alpha}', \psi')$ is represented by a local analytic isomorphism $\eta : (\mathrm{T}', \mathrm{t}_0') \to (\mathrm{T}, \mathrm{t}_0)$ and a morphism

$$\mathrm{f}\colon ((\eta \times \mathrm{id}_{\mathrm{X}})^* \tilde{\mathrm{V}}, (\eta \times \mathrm{id}_{\mathrm{X}})^* \tilde{\alpha}) \to ((\eta \times \mathrm{id}_{\mathrm{X}})^* \tilde{\mathrm{V}}', (\eta \times \mathrm{id}_{\mathrm{X}})^* \tilde{\alpha}')$$

of framed vector sheaves with $(\eta \times \mathrm{id}_{\mathrm{X}})^* \psi' \circ (\mathrm{f}|_{\mathrm{s}_0 \times \mathrm{X}}) = (\eta \times \mathrm{id}_{\mathrm{X}})^* \psi$.

We denote with $\mathcal{V}ec(\mathrm{X}, \mathrm{Y}, \mathrm{W})$ the functor

$$(\text{analytic spaces}) \longrightarrow (\text{sets})$$

defined by

$$\mathrm{T} \longrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of families of} \\ \text{framed vector sheaves to the data} \\ (\mathrm{X}, \mathrm{Y}, \mathrm{W}) \text{ parametrized by } \mathrm{T} \end{array} \right\}$$

and with $\mathrm{Def}(\mathrm{V}, \alpha)$ the functor

$$(\text{germs of analytic spaces}) \longrightarrow (\text{sets})$$

defined by

$$(\mathrm{T}, \mathrm{t}_0) \longrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of deformations} \\ \text{of } (\mathrm{V}, \alpha) \text{ over } (\mathrm{T}, \mathrm{t}_0) \end{array} \right\} .$$

For X compact and $\mathrm{c}_\bullet \in \mathrm{H}^{2\bullet}\mathrm{X}$ fixed cohomology classes, we let $\mathcal{V}ec(\mathrm{X}, \mathrm{Y}, \mathrm{W}, \mathrm{c}_\bullet)$ be the closed and open subfunctor of $\mathcal{V}ec(\mathrm{X}, \mathrm{Y}, \mathrm{W})$ containing all the points (V, α) with Chern classes $\mathrm{c}_\bullet \mathrm{V} = \mathrm{c}_\bullet$.

Let $\mathbb{C}[\varepsilon]$ be the $\mathbb{C}$-algebra $\mathbb{C}[\mathrm{X}]/(\mathrm{X}^2)$ with $\varepsilon = \mathrm{X} \bmod \mathrm{X}^2$. According to Schlessinger [37] or M. Artin ([1], [2]), $\mathrm{Def}(\mathrm{V}, \alpha)(\mathrm{Spec}\, \mathbb{C}[\varepsilon])$, the *infinitesimal deformations* of the framed vector sheaf (V, α), is the formal tangent space of $\mathcal{V}ec(\mathrm{X}, \mathrm{Y}, \mathrm{W})$ at the point $(\mathrm{V}, \alpha) \in \mathcal{V}ec(\mathrm{X}, \mathrm{Y}, \mathrm{W})(\mathrm{Spec}\, \mathbb{C})$. Due to [33, Lemma 1.1], $\mathrm{Def}(\mathrm{V}, \alpha)(\mathrm{Spec}\, \mathbb{C}[\varepsilon])$ carries a natural structure as a complex vector space. $\mathrm{Aut}(\mathrm{V}, \alpha)$ acts on $\mathrm{Def}(\mathrm{V}, \alpha)$ by $\mathrm{g} \cdot (\tilde{\mathrm{V}}, \tilde{\alpha}, \psi) = (\tilde{\mathrm{V}}, \tilde{\alpha}, \mathrm{g} \circ \psi)$. If this action is trivial and if there is a local moduli $\mathcal{M}$ for (V, α), then we have canonically $\mathrm{T}_{(\mathrm{V}, \alpha)}\mathcal{M} = \mathrm{Def}(\mathrm{V}, \alpha)(\mathbb{C}[\varepsilon])$.

Theorem 1.1 *For* $(\mathrm{V}, \alpha) \in \mathcal{V}ec(\mathrm{X}, \mathrm{Y}, \mathrm{W})(\operatorname{Spec} \mathbb{C})$*, there is a canonical isomorphism*

$$\operatorname{Ext}^1_{\mathrm{X}}(\mathrm{V}, \mathrm{V} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{J}_{\mathrm{Y}|\mathrm{X}}) = \operatorname{Def}(\mathrm{V}, \alpha)(\operatorname{Spec} \mathbb{C}[\varepsilon]).$$

Proof. [33, Theorem 1.2] □

A functor F on Artin rings is *formally smooth* if for small extensions $\mathrm{A} \to \bar{\mathrm{A}}$ the induced map $\mathrm{F}(\mathrm{A}) \to \mathrm{F}(\bar{\mathrm{A}})$ is surjective. If the functor is representable, then formal smoothness implies the smoothness of the local moduli.

Theorem 1.2 *Let* (V, α) *be in* $\mathcal{V}ec(\mathrm{X}, \mathrm{Y}, \mathrm{W})(\operatorname{Spec} \mathbb{C})$*. If* $\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathrm{X}}(\mathrm{V}, \mathcal{J}_{\mathrm{Y}|\mathrm{X}}\mathrm{V}) < \infty$ *and* $\operatorname{Ext}^2_{\mathrm{X}}(\mathrm{V}, \mathcal{J}_{\mathrm{Y}|\mathrm{X}}\mathrm{V}) = 0$*, then* $\operatorname{Def}(\mathrm{V}, \alpha)$ *is formally smooth.*

Proof. [33, Theorem 1.4] □

All considerations so far can be translated to the category of algebraic spaces over a field k and are valid there as well. For Theorem 1.2 to hold, we need the characteristic of k to be 0. Further sources for the theory of framed vector sheaves are the works of M. Lübke [32] in the analytic case and of M. Lehn [31] in the algebraic setup. An important result of M. Lehn is

Theorem 1.3 ([31, Theorem 3.4.1]) *Let* D' *be an effective big and nef divisor on a smooth projective surface* S*. If* D *is an effective divisor with* $\operatorname{supp} \mathrm{D}' \subset \operatorname{supp} \mathrm{D}$*,* W *a vector sheaf on* D*,* $\mathrm{c}_\bullet$ *fixed Chern classes and if* $\mathrm{H}^0(\mathrm{S}, \mathcal{H}om(\mathrm{V}, \mathrm{V}(-\mathrm{D})))$ *vanishes for all framed vector sheaves* (V, α) *in* $\mathcal{V}ec(\mathrm{S}, \mathrm{D}, \mathrm{W}, \mathrm{c}_\bullet)(\operatorname{Spec} \mathrm{k})$*, then* $\mathcal{V}ec(\mathrm{S}, \mathrm{D}, \mathrm{W}, \mathrm{c}_\bullet)$ *is represented by a separated algebraic space of finite type.*

We recall that a divisor is *big and nef*, if its self-intersection is positive and its intersection with any integral curve C on S is non-negative.

D. Huybrechts and M. Lehn consider in [20] the more general notion of *framed modules* over an algebraic nonsingular projective variety X as given by a coherent $\mathcal{O}_{\mathrm{X}}$-module V and a morphism from V to a fixed coherent $\mathcal{O}_{\mathrm{X}}$-module D. They introduce a notion of stability in this situation and show the existence of fine moduli spaces for the stable objects. Moreover, they prove

Theorem 1.4 ([20, Section 4]) *The formal tangent space of a framed torsion free module* $(\mathrm{V}, \mathrm{V} \to \mathrm{D})$ *is canonically isomorphic to* $\mathbb{E}\mathrm{xt}^1(\mathrm{V}, \mathrm{V} \to \mathrm{D})$. *Its deformation functor is formally smooth if* $\mathbb{E}\mathrm{xt}^2(\mathrm{V}, \mathrm{V} \to \mathrm{D})$ *vanishes.*

Here, we think of a sheaf as a complex of sheaves concentrated at zero and of a morphism of sheaves as a complex of sheaves concentrated at zero and one (see [14] for a quick reference to hyper-ext groups). In the case where a framed module is actually a framed vector sheaf in the above introduced sense, Theorem 1.4 is equivalent to Theorems 1.1 and 1.2 applied to the smooth projective case.

1.2 Framed physical instanton bundles

We consider the twistor fibration $\pi : \mathrm{P} \to \mathrm{M}$ over a real four dimensional compact manifold M with self-dual Riemannian metric. P is a three dimensional complex manifold with an induced antiholomorphic fixpoint free involution τ, an antipodal map on the twistor fibers (cf. [3, 6, 9]). A *line* on P is a complex submanifold $\mathrm{L} \subset \mathrm{P}$ with $\mathrm{L} \cong \mathbb{P}^1_{\mathbb{C}}$ and normal bundle $\mathcal{N}_{\mathrm{L}|\mathrm{P}} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. In particular, twistor fibres are lines. We denote with $\mu : \mathrm{Z} \to \mathrm{H}$ the universal Douady family of lines in P. The involution τ maps lines to lines and consequently induces an antiholomorphic involution on H. Then M appears as a set of fixpoints of τ and moreover as a real-analytic submanifold of H (cf. [3, 29]):

$$
\begin{array}{ccccc}
 & & \mathrm{H} \times \mathrm{P} & & \\
 & & \cup & & \\
\mathrm{P} = \mathrm{Z} \times_{\mathrm{H}} \mathrm{M} & \longrightarrow & \mathrm{Z} & \xrightarrow{\nu} & \mathrm{P} \\
\downarrow^{\pi} & & \downarrow^{\mu} & & \\
\mathrm{M} & \subset & \mathrm{H} & &
\end{array}
$$

The *degree* of a divisor on P means the degree of the restriction of the corresponding line bundle to a twistor fibre. Let $\mathrm{S}' \subset \mathrm{P}$ be a surface of degree 1 containing a twistor fibre F'. By [27, Proposition 2.1], S' is a smooth algebraic surface and F' the only twistor fibre in S'. With $\bar{\mathrm{S}}' = \tau(\mathrm{S})$ we have $\mathrm{F}' = \mathrm{S}' \cdot \bar{\mathrm{S}}'$. The linear system $|\mathrm{F}'|$ defines a birational morphism $\mathrm{S}' \to \mathbb{P}^2_{\mathbb{C}}$, and we have $\mathrm{M} \approx \overset{n}{\#} (-\mathbb{P}^2_{\mathbb{C}})$ or S^4. A well-examined class of examples are the LeBrun twistor spaces which fulfill the additional property $\dim|\mathrm{S}| \geq 1$ (cf. [30, 27]). These twistor spaces are classified as modifications of conic bundles and are algebraic in the sense of M. Artin [25]; i.e. they are Moishezon spaces.

A *(mathematical) instanton bundle* is an holomorphic vector bundle on P, trivial on all twistor fibres. The Penrose-Ward transformation gives us an analytic equivalence between the categories of instanton bundles and of pairs (E, ∇) of complex vector bundles on M with self-dual connection. The pair (E, ∇) is associated to the C^∞-bundle $\pi^*\mathrm{E}$ together with the holomorphic structure defined by $\bar{\partial} = (\pi^*\nabla)^{01}$. Conversely, an instanton bundle V gives rise to a pair (E, ∇) by taking $\mathrm{E} = (\mu_*\nu^*\mathrm{V})\,|_{\mathrm{M}}$ and ∇ as the restriction of

$$\begin{array}{ccc} \mu_* \left(\mathcal{O}_{\mathrm{Z}} \otimes_{\nu^{-1}\mathcal{O}_{\mathrm{P}}} \nu^{-1}\mathrm{V}\right) & \xrightarrow{\mu_*(\mathrm{d}_{\mathrm{Z}|\mathrm{P}} \otimes \mathrm{id}_{\nu^{-1}\mathrm{V}})} & \mu_*(\Omega^1_{\mathrm{Z}|\mathrm{P}} \otimes_{\mathcal{O}_{\mathrm{Z}}} \nu^*\mathrm{V}) \\ \downarrow \nabla & & \downarrow \cong \\ \Omega^1_{\mathrm{H}} \otimes_{\mathcal{O}_{\mathrm{H}}} \mu_*\nu^*\mathrm{V} & \xrightarrow{\cong} & \mu_*\Omega^1_{\mathrm{Z}|\mathrm{P}} \otimes_{\mathcal{O}_{\mathrm{H}}} \mu_*\nu^*\mathrm{V} \end{array}$$

(cf. [3, 29]).

For G a linear group as, for example, U(r), SU(r), Sp(r) or SO(r), a G-*instanton* on M is given by a complex G-vector bundle E with self-dual connection ∇ compatible with the G-structure on E. The Penrose-Ward transformation associates a G-instanton to an instanton bundle with additional properties. In particular, the category of U(r)-instantons is analytically equivalent to the category of instanton bundles V on P for which there is an isomorphism $\varphi : \mathrm{V} \cong \tau^*\bar{\mathrm{V}}^\vee$ with $\tau^*\bar{\varphi}^\vee = \varphi$, where $\bar{\mathrm{V}}^\vee$ denotes the bundle of antilinear forms. We denote these instanton bundles as U(r)*-instanton bundles or as physical instanton bundles* (cf. [3, 6, 29]). We are especially interested in the subcase of SU(r)*-instanton bundles* for which we have to add the condition $\det \mathrm{V} \cong \mathcal{O}_{\mathrm{P}}$ or equivalently, $\mathrm{c}_1\mathrm{V} = 0$.

For an instanton bundle V on P, we can fix a trivialisation $\alpha : \mathrm{V}|_{\mathrm{F}'} \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{F}'}$ along our twistor fibre $\mathrm{F}' = \mathrm{S}' \cdot \bar{\mathrm{S}}'$. The resulting pair (V, α) is called a framed instanton bundle. The frame lives in codimension 2. The corresponding notion over the four-manifold M is the *based instanton.* By restricting V to S', we obtain a framed vector bundle on a smooth rational surface, framed along a divisor that is big and nef. For this case, the moduli problem has been well-examined by Lehn and Huybrechts in the algebraic-projective case ([31, 20]) and by Lübke [32] from the analytic point of view. It was an idea of Donaldson [8] to use this restriction map to discuss moduli of framed instantons. By the results of Buchdahl [7], it follows that there is a bijection between the isomorphism classes of framed U(r)-instantons and framed vector bundles on S'.

In [33], we have described the moduli space of framed U(r)-instanton bundles $\mathrm{Ins}^{\mathrm{U(r)}}(\mathrm{P},\mathrm{F}',\mathcal{O}_{\mathrm{F}'}^{\mathrm{r}})$ as a real structure on an open part of $\mathcal{V}ec(\mathrm{P},\mathrm{F}',\mathcal{O}_{\mathrm{F}'}^{\mathrm{r}})$ and have shown that $\mathrm{Ins}^{\mathrm{U(r)}}(\mathrm{P},\mathrm{F}',\mathcal{O}_{\mathrm{F}}^{\mathrm{r}})$ is real-analytically isomorphic to $\mathcal{V}ec(\mathrm{S}',\mathrm{F}',\mathcal{O}_{\mathrm{F}'}^{\mathrm{r}})$, which is a representable moduli functor due to Theorem 1.3. In particular, we have

Theorem 1.5 *The space $\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r})$ of based* SU(r)*-instantons of charge* m *and rank* r *is real-analytically isomorphic to* $\mathcal{V}ec(\mathrm{S}',\mathrm{F}',\mathcal{O}_{\mathrm{F}'}^{\mathrm{r}},\mathrm{c}_1=0,\mathrm{c}_2=\mathrm{m})$.

Throughout the rest of the paper, we will denote $\mathcal{V}ec(\mathrm{S}',\mathrm{F}',\mathcal{O}_{\mathrm{F}'}^{\mathrm{r}},\mathrm{c}_1=0,\mathrm{c}_2=\mathrm{m})$ with $\mathcal{I}(\mathrm{m},\mathrm{r})$.

1.3 Vector bundles on blown up ruled surfaces

We consider a smooth projective curve N of genus g over Spec k, where, from now on, we assume that k is a field of characteristic zero. We fix a closed point $\infty \in \mathrm{N}$. Now let $\pi : \mathrm{S} \to \mathrm{N}$ be the blowing up of a ruled surface over N via n successive monoidal transformations with centres in fibres over $\mathrm{N} - \{\infty\}$, and let $\mathrm{N} \subset \mathrm{S}$ be a section of π disjoint to the exceptional locus. For points P on N, we denote the fibre of π over P with F_{P}. For η the generic point on N, we have in particular F_η as the generic fibre of π.

We define $\mathcal{V}'(\mathrm{S},\mathrm{m},\mathrm{r})$ as the open subfunctor of $\mathcal{V}ec(\mathrm{S},\mathrm{N},\mathcal{O}_{\mathrm{F}}^{\mathrm{r}},\mathrm{c}_1=0,\mathrm{c}_2=\mathrm{m})$, which associates to an algebraic space T the isomorphism classes of all those families (V,α) where $\mathrm{V}|_{\mathrm{T}\times\mathrm{F}_\eta}$ is free. Furthermore, we define $\mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r})$ as the open subfunctor of $\mathcal{V}'(\mathrm{S},\mathrm{m},\mathrm{r})$ which associates to an algebraic space T the isomorphism classes of all those families (V,α) where $\mathrm{V}|_{\mathrm{F}_\infty}$ is free. If the context is clear, we just write $\mathcal{V}'(\mathrm{m},\mathrm{r})$ and $\mathcal{V}(\mathrm{m},\mathrm{r})$ instead of $\mathcal{V}'(\mathrm{S},\mathrm{m},\mathrm{r})$ or $\mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r})$.

Theorem 1.6 *The moduli functors* $\mathcal{V}'(\mathrm{m},\mathrm{r})$ *and* $\mathcal{V}(\mathrm{m},\mathrm{r})$ *are represented by fine and smooth moduli spaces of dimension* $2\mathrm{rm}$.

Proof. For a vector sheaf which is free along F_∞, a framing along N induces a framing along $\mathrm{N}\cup\mathrm{F}_\infty$. Therefore, the functor $\mathcal{V}(\mathrm{m},\mathrm{r})$ is naturally equivalent to $\mathcal{V}ec(\mathrm{S},\mathrm{N}\cup\mathrm{F}_\infty,\mathcal{O}^{\mathrm{r}}_{\mathrm{N}\cup\mathrm{F}_\infty},\mathrm{c}_1=0,\mathrm{c}_2=\mathrm{m})$. The divisor $\mathrm{N}+\mathrm{F}_\infty$ has the same support as the effective divisor $\mathrm{N}+\max(1,-\mathrm{N}\cdot\mathrm{N})\mathrm{F}_\infty$, which is big and nef as a corollary of [13, Proposition V.2.3 and Proposition V.3.2].

For (V,α) a point in $\mathcal{V}(\mathrm{m},\mathrm{r})$, we have V as trivial on general fibres of π. The same property is true for the vector sheaf $\mathcal{E}nd\,\mathrm{V}$, and we obtain the vanishing of $\mathrm{H}^0(\mathrm{U},(\mathcal{E}nd\,\mathrm{V})(-\mathrm{N}-\mathrm{F}_\infty)$, where U is the open neighbourhood of S containing all fibres of π where $\mathcal{E}nd\,\mathrm{V}$ is trivial. Since $\mathcal{E}nd\,\mathrm{V}$ is torsion free and since S is irreducible, the long exact sequence of cohomology with support in $\mathrm{S}-\mathrm{U}$ shows that $\mathrm{H}^0(\mathrm{S},(\mathcal{E}nd\,\mathrm{V})(-\mathrm{N}-\mathrm{F}_\infty)$ vanishes, too. Thus, we can apply Theorem 1.3 to obtain the existence of a fine moduli space for $\mathcal{V}(\mathrm{m},\mathrm{r})$.

Again, because of the triviality of points (V,α) in $\mathcal{V}(\mathrm{m},\mathrm{r})$ along general fibres of π, we have $\mathrm{Hom}(\mathrm{V},\mathrm{V}(-\mathrm{N}))=0$. By Serre duality, we have $\mathrm{Ext}^2(\mathrm{V},\mathrm{V}(-\mathrm{N}))$ equal to $\mathrm{Hom}(\mathrm{V},\mathrm{V}(\mathrm{N})\otimes\omega)^\vee$, where ω denotes the dualising sheaf on S. Because of [13, Lemma V.2.10], ω has degree -2 on fibres of π. Hence, for general fibres F of π, we have $\mathrm{Hom}(\mathrm{V}|_\mathrm{F},\mathrm{V}|_\mathrm{F}(\mathrm{N})\otimes\omega|_\mathrm{F})=0$, and, analogously as above, we infer the vanishing of $\mathrm{Ext}^2(\mathrm{V},\mathrm{V}(-\mathrm{N}))=\mathrm{Hom}(\mathrm{V},\mathrm{V}(\mathrm{N})\otimes\omega)^\vee$. By Theorem 1.2, this shows the smoothness of $\mathcal{V}(\mathrm{m},\mathrm{r})$.

We have also obtained that the dimension of $\mathcal{V}(\mathrm{m},\mathrm{r})$, which equals the dimension of the vector space $\mathrm{Ext}^1(\mathrm{V},\mathrm{V}(-\mathrm{N}))$, equals the Euler characteristic $-\chi(\mathcal{E}nd\,\mathrm{V})(-\mathrm{N})$. By an easy computation, Hirzebruch-Riemann-Roch shows this number to be $2\mathrm{rm}$.

The result for $\mathcal{V}(\mathrm{m},\mathrm{r})$ can be easily extended to $\mathcal{V}'(\mathrm{m},\mathrm{r})$, as $\mathcal{V}'(\mathrm{m},\mathrm{r})$ has an open covering $\{\mathcal{V}_\mathrm{P}(\mathrm{m},\mathrm{r})|\mathrm{P}\in\mathrm{N}\}$, where $\mathcal{V}_\mathrm{P}(\mathrm{m},\mathrm{r})$ is the open subfunctor of $\mathcal{V}'(\mathrm{m},\mathrm{r})$

which associates to an algebraic space T the isomorphism classes of all those families (V, α) for which $\mathrm{V}|_{\mathrm{F_P}}$ is free. All these open subfunctors are represented by smooth algebraic spaces for exactly the same reasons as for $\mathcal{V}(\mathrm{m},\mathrm{r}) = \mathcal{V}_\infty(\mathrm{m},\mathrm{r})$. Thus, $\mathcal{V}'(\mathrm{m},\mathrm{r})$ is represented by a smooth algebraic space, too. $\square$

In this proof, we have intensively used the fact that the triviality of a vector sheaf along one fibre of π implies the triviality along general fibres. This fact is due to the rigidity of the trivial bundle on the projective line. One consequence is that, for (V, α) a point in $\mathcal{V}(\mathrm{m},\mathrm{r})$, V is non-trivial along only finitely many fibres $\mathrm{F}_{\mathrm{P}_1}, \ldots, \mathrm{F}_{\mathrm{P}_\mathrm{q}}$ of π. These are the *jumping fibres* of (V, α). They are important continuous invariants of our framed vector sheaves. This statement can be made more precise in the following way:

Let $\mathrm{p} : \mathrm{C} \to \mathrm{B}$ be a curve over B (i.e., a flat and projective morphism of relative dimension 1, where B is supposed to be irreducible), and let $\mathrm{D} \subset \mathrm{C}$ be an effective divisor in C which is finite over B and relatively ample with respect to p. Let W be a vector sheaf on C with the properties that the Euler characteristic of W restricted to fibres of p vanishes, and that there is at least one fibre where the cohomology vanishes as well. Therefore, we have $\mathrm{p}_*\mathrm{W} = 0$ as well as $\mathrm{R}^1\mathrm{p}_*\mathrm{W}(\ell\mathrm{D}) = 0$ for ℓ big enough. Hence, there is the short exact sequence

$$0 \to \mathrm{p}_*\mathrm{W}(\ell\mathrm{D}) \xrightarrow{\mathrm{s}} \mathrm{p}_*(\mathrm{W} \otimes (\mathcal{O}_\mathrm{C}(\ell\mathrm{D})/\mathcal{O}_\mathrm{C})) \longrightarrow \mathrm{R}^1\mathrm{p}_*\mathrm{W} \to 0.$$

Because of the assumptions, the first two modules in this sequence are locally free, and we define the divisor Δ of the determinant of s to be the *determinantal divisor* of W with respect to p. This Δ does not depend on the choices made. This concept is well-known and well-used in the theory of vector sheaves on curves and their moduli.

We note that for (V, α) an element in some $\mathcal{V}(\mathrm{m},\mathrm{r})(\mathrm{T})$, we may apply the concept of determinantal divisors to the sheaves $\mathrm{V}(-(\mathrm{T} \times \mathrm{N}))$ with respect to the curve $\mathrm{T} \times \mathrm{S} \to \mathrm{T} \times \mathrm{N}$.

Theorem 1.7 *Due to the natural map, which assigns to a family of framed vector sheaves* (V, α) *in* $\mathcal{V}(\mathrm{m},\mathrm{r})(\mathrm{T})$ *the determinantal divisor associated to* $\mathrm{V}(-(\mathrm{T} \times \mathrm{N}))$*,* *there is a morphism*

$$\Delta : \mathcal{V}'(\mathrm{m}, \mathrm{r}) \to \mathrm{Sym}^{\mathrm{m}}(\mathrm{N})$$

with restriction

$$\Delta : \mathcal{V}(\mathrm{m}, \mathrm{r}) \to \mathrm{Sym}^{\mathrm{m}}(\mathrm{N} - \{\infty\}).$$

Proof. The number m is obtained by an easy computation via Grothendieck-Riemann-Roch. □

What are the fibres of Δ? For P a point on N, we define $\mathrm{S_P}$ to be the localisation of S around $\mathrm{F_P}$. The restrictions of π and N to $\mathrm{S_P}$ are again denoted with π and N. We call V *a jump on* $\mathrm{S_P}$ if V is a vector sheaf on $\mathrm{S_P}$ which is trivial along F_η and which has trivial determinant. For such a given jump V, there is a determinantal divisor associated to $\mathrm{V}(-\mathrm{N})$ as above. This determinantal divisor is of the form $\mathrm{m} \cdot \mathrm{P}$, where m equals the length of the module $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N})$, and is called *the order of* V. We put $\mathcal{J}(\mathrm{S_P}, \mathrm{m}, \mathrm{r})$ to be the set of isomorphism classes of all framed jumps on $\mathrm{S_P}$; i.e., of all pairs (V, α) with V a jump of rank r and order m and α a framing $\mathrm{V}|_{\mathrm{N}} \cong \mathcal{O}_{\mathrm{N}}^{\mathrm{r}}$.

Theorem 1.8 *(Cutting and gluing) Via restriction, the set of closed points in the inverse image* $\Delta^{-1}(\mathrm{m_1P_1} + \ldots + \mathrm{m_qP_q})$ *is naturally bijective to*

$$\mathcal{J}(\mathrm{S}_{\mathrm{P}_1}, \mathrm{m}_1, \mathrm{r}) \times \ldots \times \mathcal{J}(\mathrm{S}_{\mathrm{P}_\mathrm{q}}, \mathrm{m}_\mathrm{q}, \mathrm{r}).$$

Proof. The map

$$\Delta^{-1}(\mathrm{m_1P_1} + \ldots + \mathrm{m_qP_q}) \to \mathcal{J}(\mathrm{S}_{\mathrm{P}_1}, \mathrm{m}_1, \mathrm{r}) \times \ldots \times \mathcal{J}(\mathrm{S}_{\mathrm{P}_\mathrm{q}}, \mathrm{m}_\mathrm{q}, \mathrm{r})$$

is given by mapping a closed point (V, α) to the q-tuple $((\mathrm{V}_1, \alpha_1), \ldots, (\mathrm{V_q}, \alpha_\mathrm{q}))$, where $(\mathrm{V_i}, \alpha_\mathrm{i})$ is the restriction of (V, α) to $\mathrm{S}_{\mathrm{P_i}}$. The inverse map is obtained as follows:

Since $\mathrm{V_i}$ is trivial along F_η, the framing α_i induces a framing $\mathrm{V_i}|_{\mathrm{F}_\eta} \cong \mathcal{O}_{\mathrm{F}_\eta}^{\mathrm{r}}$. We also have a canonical isomorphism $\mathcal{O}^{\mathrm{r}}_{\mathrm{S}-\mathrm{F}_1-\ldots-\mathrm{F_q}}|_{\mathrm{F}_\eta} \cong \mathcal{O}_{\mathrm{F}_\eta}^{\mathrm{r}}$. Therefore,

$$\mathrm{V} = \mathcal{O}^{\mathrm{r}}_{\mathrm{S}-\mathrm{F}_1-\ldots-\mathrm{F_q}} \sqcup_{\mathcal{O}^{\mathrm{r}}_{\mathrm{F}_\eta}} \mathrm{V}_1 \sqcup_{\mathcal{O}^{\mathrm{r}}_{\mathrm{F}_\eta}} \ldots \sqcup_{\mathcal{O}^{\mathrm{r}}_{\mathrm{F}_\eta}} \mathrm{V_q}$$

is a well-defined vector sheaf on S, and is indeed the same V as we have started with. Moreover, we recover the original α.$\square$

From Theorem 1.8 we infer that the set $\mathcal{J}(\mathrm{S_P}, \mathrm{m}, \mathrm{r})$ is naturally bijective to the closed points of the closed subfunctor of $\mathcal{V}(\mathrm{m}, \mathrm{r})$ of those families whose image under Δ is the constant family of divisors mP. Thus, $\mathcal{J}(\mathrm{S_P}, \mathrm{m}, \mathrm{r})$ itself can be considered as a functor. Since this functor is given as a fibre of a morphism between algebraic spaces, it is represented by an algebraic space, too.

If (V_1, α_1) and (V_2, α_2) are elements in $\mathcal{V}'(\mathrm{m}_1, \mathrm{r})(\operatorname{Spec}\mathrm{k})$ and $\mathcal{V}'(\mathrm{m}_2, \mathrm{r})(\operatorname{Spec}\mathrm{k})$ with different jumping lines, then we may associate to them an element (V, α) in $\mathcal{V}'(\mathrm{m}_1 + \mathrm{m}_2, \mathrm{r})(\operatorname{Spec}\mathrm{k})$ by the previously described cutting-and-gluing procedure. In particular, if (V_1, α_1) in $\mathcal{V}'(\mathrm{m}_1, \mathrm{r})(\operatorname{Spec}\mathrm{k})$ is trivial along $\mathrm{F_P}$ and (V_2, α_2) is an element in $\mathcal{J}(\mathrm{S_P}, \mathrm{m}_2, \mathrm{r})(\operatorname{Spec}\mathrm{k})$, then we may *add* the framed jump (V_2, α_2) to the framed vector sheaf (V_1, α_1) to obtain an element in $\mathcal{V}'(\mathrm{m}_1 + \mathrm{m}_2, \mathrm{r})(\operatorname{Spec}\mathrm{k})$.

We could reformulate now Theorem 1.8 for families instead of closed points. For $\mathrm{k} = \mathbb{C}$, we may translate the whole situation into the complex analytic setup. In Section 3, we will make use of the following

Corollary 1.9 *Consider an analytic space* B *and a family* (V, α) *in* $\mathcal{V}(\mathrm{m}, \mathrm{r})(\mathrm{B})$, *whose image under* Δ *is* $\mathrm{D}_0 \in \mathrm{Sym}^{\mathrm{m}}\mathrm{N}(\mathrm{B})$. *Assume, the points* $\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{n}$ *in* $\mathrm{N} - \{\infty\}$ *are fixed in such a way, that we can choose an isomorphism*

$$\pi^{-1}(\mathrm{N} - \{\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{n}\}) \cong (\mathrm{N} - \{\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{n}\}) \times \mathbb{P}^1_{\mathbb{C}}.$$

Let $(\mathrm{T}, \mathrm{t}_0)$ *be a based topological space and* $\mathrm{D} : \mathrm{T} \times \mathrm{B} \to \mathrm{Sym}^{\mathrm{m}}(\mathrm{N} - \{\infty\})$ *be a continuous map such that*

(i) $\mathrm{D}|_{\{\mathrm{t}_0\} \times \mathrm{B}}$ *equals the map* $\mathrm{B} \to \mathrm{Sym}^{\mathrm{m}}(\mathrm{N} - \{\infty\})$ *induced by* D_0*;*

(ii) and, for $\mathrm{D}(\mathrm{t}, \mathrm{b}) = \mathrm{m}_1\mathrm{P}_1 + \ldots + \mathrm{m}_\mathrm{q}\mathrm{P}_\mathrm{q} + \mathrm{n}_1\mathrm{Q}_1 + \ldots + \mathrm{n}_\mathrm{n}\mathrm{Q}_\mathrm{n}$ *with* $\mathrm{P}_\mathrm{i} \neq \mathrm{P}_\mathrm{j}$ *for* $\mathrm{i} \neq \mathrm{j}$ *and* $\mathrm{P}_\mathrm{i} \neq \mathrm{Q}_\mathrm{j}$ *for all* i, j, *the integers* $\mathrm{q}, \mathrm{m}_1, \ldots, \mathrm{m}_\mathrm{q}, \mathrm{n}_1, \ldots, \mathrm{n}_\mathrm{n}$ *do not depend on* t.

Via cutting and gluing, there is a unique lift $\tilde{\mathrm{D}} : \mathrm{T} \times \mathrm{B} \to \mathcal{V}(\mathrm{m}, \mathrm{r})$ *of* D *along* Δ *such that* $\tilde{\mathrm{D}}|_{\{\mathrm{t}_0\} \times \mathrm{B}}$ *is equal to the map* $\mathrm{B} \to \mathcal{V}(\mathrm{m}, \mathrm{r})$ *induced by* (V, α).

Proof. A fixed isomorphism

$$\pi^{-1}(\mathrm{N} - \{\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{n}\}) \cong (\mathrm{N} - \{\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{n}\}) \times \mathbb{P}^1_{\mathbb{C}}$$

induces a fixed isomorphism

$$\mathcal{J}(\mathrm{S}_\mathrm{P}, \mathrm{m}, \mathrm{r}) \cong \mathcal{J}(\mathrm{S}_{\mathrm{P}_\infty}, \mathrm{m}, \mathrm{r})$$

for all $\mathrm{P} \in \mathrm{N} - \{\mathrm{Q}_1, \ldots, \mathrm{Q}_\mathrm{n}\}$. Let $(\mathrm{V}_\mathrm{b}, \alpha_\mathrm{b})$ be the restriction of (V, α) to $\{\mathrm{b}\} \times \mathrm{S}$, and consider $\mathrm{D}(\mathrm{t}, \mathrm{b}) = \mathrm{m}_1\mathrm{P}_1(\mathrm{t}) + \ldots + \mathrm{m}_\mathrm{q}\mathrm{P}(\mathrm{t})_\mathrm{q} + \mathrm{n}_1\mathrm{Q}_1 + \ldots + \mathrm{n}_\mathrm{n}\mathrm{Q}_\mathrm{n}$ with $\mathrm{P}_\mathrm{i} \neq \mathrm{P}_\mathrm{j}$. By Theorem 1.8, $(\mathrm{V}_\mathrm{b}, \alpha_\mathrm{b})$ is given by certain framed jumps $(\mathrm{V}_{\mathrm{b}1}, \alpha_{\mathrm{b}1})$, ..., $(\mathrm{V}_{\mathrm{bq}}, \alpha_{\mathrm{bq}})$ with $(\mathrm{V}_{\mathrm{bi}}, \alpha_{\mathrm{bi}}) \in \mathcal{J}(\mathrm{S}_{\mathrm{P}_\mathrm{i}}, \mathrm{m}_\mathrm{i}, \mathrm{r}) \cong \mathcal{J}(\mathrm{S}_{\mathrm{P}_\infty}, \mathrm{m}_\mathrm{i}, \mathrm{r})$. We obtain $\tilde{\mathrm{D}}(\mathrm{t}, \mathrm{b})$ by adding these framed jumps to the framed trivial vector sheaf at the fibres over $\mathrm{P}_1(\mathrm{t}), \ldots,\ \mathrm{P}_\mathrm{q}(\mathrm{t})$, respectively. □

Now we explain what these framed vector sheaves on blown up, ruled surfaces have to do with the moduli $\mathcal{I}(\mathrm{m}, \mathrm{r})$ of framed SU(r)-instanton bundles introduced in Section 1.2.

We may think about the smooth rational surface S' as being obtained from $\mathbb{P}^2_\mathrm{k}$ via n successive monoidal transformations. Indeed, if P is a LeBrun twistor space, then S' is the blowing up of the projective plane in n different points, situated all on one line. We continue with two different approaches, which in fact look quite alike.

First, we choose a point P' on F', consider the blowing up $\mathrm{S} \to \mathrm{S}'$ in P' with exceptional divisor N, and obtain a curve $\pi : \mathrm{S} \to \mathrm{N}$ with section over a base, again denoted with N, which looks like the n-fold blowing up of the first Hirzebruch surface. We choose the point $\infty \in \mathrm{N}$ such that F_∞ is the strict transform of F'.

In the second approach, we choose two different points P' and Q' on F' and define S as the surface obtained by blowing up S' in both points and contracting the strict transform of F' afterwards. We denote with N and F_∞ the images of the exceptional divisors over P' and Q', respectively. We have thus obtained two curves $\pi : \mathrm{S} \to \mathrm{N}$ with section N and $\kappa : \mathrm{S} \to \mathrm{F}$ with section F_∞, which are both isomorphic to the n-fold blowing up of $\mathbb{P}^1_\mathrm{k} \times \mathbb{P}^1_\mathrm{k} \to \mathbb{P}^1_\mathrm{k}$.

In both approaches, $\pi : \mathrm{S} \to \mathrm{N}$ is the n-fold blowing up of a ruled surface with centre disjoint to the fixed section and a fixed fibre. Due to pullback and direct image, we obviously have

Theorem 1.10 $\mathcal{V}(\mathrm{S}, \mathrm{m}, \mathrm{r}) = \mathcal{I}(\mathrm{m}, \mathrm{r})$.

We remark that in the general case, where S' is the blowing up of $\mathbb{P}^2_{\mathrm{k}}$ in n distinct points, we can choose the point(s) P' (and Q') above in such a way that the fibres of π (and κ) in the first approach (respectively in the second approach) are either smooth rational curves or the union of two smooth rational (-1)-curves which meet transversally. This is in particular possible for LeBrun-twistor spaces and justifies why we consider in Section 2 and 3 only such general blow ups. This restriction is not made in Section 4.

2 Framed local jumps

2.1 Ordinary and exceptional local jumps

We consider a discrete valuation ring A finitely generated over a field k of characteristic zero with maximal ideal $\mathrm{m_A}$ and minimal ideal η and define $\mathrm{A_n} = \mathrm{A} \otimes \mathrm{m_A^{n+1}}$ for $\mathrm{n} \geq 0$. We let Y be the local line $\mathbb{P}^1_{\mathrm{A}} = \mathrm{Proj}\,\mathrm{A}[\mathrm{x}, \mathrm{y}] \xrightarrow{\pi} \mathrm{Spec}\,\mathrm{A}$, $\mathrm{Y_n}$ the restriction of Y over $\mathrm{Spec}\,\mathrm{A_n}$, Y_η the generic fibre of π over η and $\mathrm{N} = \mathrm{Z}_+(\mathrm{y})$ the section of π at infinity. We will frequently identify quasi coherent sheaves on Spec A and A-modules.

An *ordinary local jump* is given by a vector sheaf V on Y with trivial determinant, which is trivial along Y_η. We obtain that $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N})$ is an A-module of finite length m=ord V, the *order* or the *multiplicity* of the jump V. With $\mathrm{ord_n}\mathrm{V}$, the n*-th order* of V, we denote the length of the module $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N}) \otimes \mathrm{A_n}$. Because of base change, we have $\mathrm{ord_n V} = \dim_{\mathrm{k}} \mathrm{H}^1(\mathrm{Y_n}, \mathrm{V}|_{\mathrm{Y_n}})$. In particular, in the case $\mathrm{r} = \mathrm{rk}\,\mathrm{V} = 2$ we have $\mathrm{ord_0 V}$ as the *splitting type* of $\mathrm{V_0}$, that is $\mathrm{V}|_{\mathrm{Y_0}} = \mathcal{O}_{\mathrm{Y_0}}(\mathrm{ord_0 V}) \oplus \mathcal{O}_{\mathrm{Y_0}}(-\mathrm{ord_0 V})$. A *framed ordinary jump* consists of a pair (V, α), where V is an ordinary jump and α is a framing $\alpha : \mathrm{V}|_{\mathrm{N}} \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{N}}$.

We embed $\mathrm{Spec}\,A$ into the smooth projective curve N associated to the field of fractions of A. We fix closed points $0 = m_A \neq \infty \in N$, consider the ruled surface $S = N \times \mathbb{P}^1_k \to N$ and define $\mathcal{J}_A(m,r) = \mathcal{J}(S_0, m, r)$. That is, $\mathcal{J}_A(m,r)$ is the closed subfunctor of $\mathcal{V}(S,m,r)$ which associates to an algebraic space T only the isomorphism classes of those families (V,α) where $V|_{T\times(S-F_0)}$ is free. As a corollary of Theorem 1.8, $\mathcal{J}_A(m,r)$ is a functor represented by an algebraic space, such that $\mathcal{J}_A(m,r)(\mathrm{Spec}\,k)$ equals the set of isomorphism classes of framed ordinary jumps. We will see later that $\mathcal{J}_A(m,r)$ carries the structure of an algebraic variety. Unless it is not clear from the context, we will write $\mathcal{J}(m,r)$ instead of $\mathcal{J}_A(m,r)$ in the following.

The rational surface S' from Section 1.2 and 1.3, an n-fold blowing up of the projective plane, is in the general case the blowing up in n distinct points. As we remarked before, this is especially true for surfaces of degree 1 in LeBrun twistor spaces. We may assume that all fibres of $\pi : S \to N$ from Section 1.3 are either smooth rational curves in the general case, or unions of two (-1)-curves in exceptional cases. In other words, the local fibres S_P are either isomorphic to Y, or isomorphic to

$$X = Z(x_0x_1 - ty_0y_1) \subset \mathrm{Proj}\,A[x_0,y_0] \times_{\mathrm{Spec}\,A} \mathrm{Proj}\,A[x_0,y_0],$$

the blowing up of Y in a point on the central fibre, which will be referred to as the *exceptional local line*. We fix inhomogeneous coordinates $z_i = x_i/y_i$ for $i = 0,1$. The morphism $X \to \mathrm{Spec}\,A$ is again denoted with π.

In the following, (i,j) will always be a permutation of $(0,1)$. We fix the closed subvarieties $E_i = Z(t,x_j)$, $N = N_0 = Z(y_0)$ and $N_1 = Z(y_1)$. Corresponding to the contraction of E_1 or E_0, we obtain two monoidal transformations

$$\begin{array}{ccl} & \overset{\sigma_0}{\nearrow} & Y_0 = \mathrm{Proj}\,\mathbb{C}[t]_{(t)}[x_0,y_0] \\ X & & \\ & \underset{\sigma_1}{\searrow} & Y_1 = \mathrm{Proj}\,\mathbb{C}[t]_{(t)}[x_1,y_1] \end{array}$$

with center in the points $P_i = Z(x_i,t) \subset Y_i$ and with $\sigma_i : N_i \cong N_i$, where N_i is the section $y_i = 0$ in Y_i as well as in X.

Now let (V, α_0) be a *framed exceptional jump*, which means that V is a vector sheaf of rank r on X with trivial determinant $\det \mathrm{V} = \Lambda^{\mathrm{r}}\mathrm{V}$, such that the restriction on the generic fibre $\mathrm{V}|_{\mathrm{X}_\eta}$ is trivial, and that $\alpha_0 : \mathrm{V}|_{\mathrm{N}_0} \cong \mathcal{O}_{\mathrm{N}_0}^{\mathrm{r}}$ is a framing of V along the section N_0. We define the *order* of the exceptional jump V as the length of the module $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N}_0)$.

Lemma 2.1 *For* (V, α_0) *a framed exceptional jump, the length of* $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N}_0)$ *is equal to the length of* $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N}_1)$.

Proof. Because $\mathrm{N_i} - \mathrm{N_j} - \mathrm{E_j} = \operatorname{div} \mathrm{z_i}$, we have linear equivalence between $\mathrm{N_i} - \mathrm{N_j}$ and $\mathrm{E_j}$. From

$$0 \to \mathrm{V}(-\mathrm{N_i}) \longrightarrow \mathrm{V}(-\mathrm{N_j})(\cong \mathrm{V}(\mathrm{E_j} - \mathrm{N_i})) \longrightarrow \mathrm{V}|_{\mathrm{E_j}}(-1) \to 0$$

we obtain an exact sequence

$$0 \to \mathrm{H}^0(\mathrm{E_j}, \mathrm{V}|_{\mathrm{E_j}}(-1)) \to \mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N_i}) \to \mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N_j}) \to \mathrm{H}^1(\mathrm{E_j}, \mathrm{V}|_{\mathrm{E_j}}(-1)) \to 0.$$

Since $\det \mathrm{V} \cong \mathcal{O}_{\mathrm{X}}$, we infer $\chi(\mathrm{V}|_{\mathrm{E_j}}(-1)) = 0$. Therefore, the modules $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N_i})$ and $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N_j})$ have the same length, and the definition of the order of an exceptional jump is invariant under permutations of (0,1).□

We proceed in the same way as we have done for ordinary jumps and embed Spec A into the smooth projective curve N associated to the field of fractions of A. We fix closed points $0 = \mathrm{m_A} \neq \infty \in \mathrm{N}$, consider the blowing up $\mathrm{S} \to \mathrm{N}$ of the ruled surface $\mathrm{N} \times \mathbb{P}^1_{\mathrm{k}} \to \mathrm{N}$ in the point $(0, 0:1)$ and define $\mathcal{E}_{\mathrm{A}}(\mathrm{m}, \mathrm{r}) = \mathcal{E}(\mathrm{S}_0, \mathrm{m}, \mathrm{r})$. That is, $\mathcal{E}_{\mathrm{A}}(\mathrm{m}, \mathrm{r})$ is the closed subfunctor of $\mathcal{V}(\mathrm{S}, \mathrm{m}, \mathrm{r})$ which associates to an algebraic space T only the isomorphism classes of those families (V, α) where $\mathrm{V}|_{\mathrm{T}\times(\mathrm{S}-\mathrm{F}_0)}$ is free. By Theorems 1.6 and 1.8, $\mathcal{E}_{\mathrm{A}}(\mathrm{m}, \mathrm{r})$ is a functor represented by an algebraic space such that $\mathcal{E}_{\mathrm{A}}(\mathrm{m}, \mathrm{r})(\operatorname{Spec} \mathrm{k})$ equals the set of isomorphism classes of framed exceptional jumps. Unless the context is not clear, we will write $\mathcal{E}(\mathrm{m}, \mathrm{r})$ instead of $\mathcal{E}_{\mathrm{A}}(\mathrm{m}, \mathrm{r})$.

Generally, we see that the moduli problem of framed exceptional jumps of given order m is symmetric with respect to the two contractions σ_0 and σ_1. One obvious result is

Lemma 2.2 *Due to σ_0^* and σ_1^*, we have two open and disjoint embeddings*

$$\mathcal{J}(\mathrm{m},\mathrm{r}) \hookrightarrow \mathcal{E}(\mathrm{m},\mathrm{r}).$$

An exceptional jump (V,α) belongs to $\sigma_i^*\mathcal{J}(m,r)$ iff V is free along $\mathrm{E_j}$. In particular, the spaces $\mathcal{E}(\mathrm{m},\mathrm{r})$ are not irreducible.

2.2 Monadic description of ordinary jumps

Here, we restrict ourselves to the case $\mathrm{A} = \mathrm{k}[\mathrm{t}]_{(\mathrm{t})}$, which will be seen later as no restriction at all. We repeat the construction and reduction of monads as can be found in the literature (e.g. [18, 28]), which is based on Hulek monads. We recall that a *monad* is a short complex

$$0 \to \mathrm{M}' \xrightarrow{\varphi} \mathrm{M} \xrightarrow{\psi} \mathrm{M}'' \to 0$$

of vector sheaves, where ψ and φ are bundle morphisms (not just sheaf morphisms), and where the only non-vanishing cohomology is the vector sheaf $\ker(\psi)/\mathrm{im}(\varphi)$.

Theorem 2.3 (K. Hulek) *If V is a vector sheaf on $\mathbb{P}^2_{\mathrm{k}}$ of rank r and with second Chern class m, which is trivial along one line, then V appears as the cohomology of a monad*

$$0 \to \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(-1) \otimes \mathrm{C}' \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}} \otimes \mathrm{C} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(1) \otimes \mathrm{C}'' \to 0,$$

where C' and C'' are k-vector spaces of dimension m and where C is a k-vector space of dimension $2\mathrm{m}+\mathrm{r}$.

Proof. We fix a line F on the plane such that $\mathrm{V}|_{\mathrm{F}}$ is free. As a consequence, we have $\mathrm{H}^0(\mathrm{V}(-\ell)) = 0$ for all positive ℓ. With Serre duality, we also obtain $\mathrm{H}^2(\mathrm{V}(-\ell)) = 0$ for all $\ell \geq -2$. Therefore, the vector spaces

$$\mathrm{C}' = \mathrm{Ext}^1(\mathrm{V}, \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(-1))^\vee$$

and

$$\mathrm{C}'' = \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(1), \mathrm{V})$$

are both of dimension m, as can be easily computed with Hirzebruch-Riemann-Roch.

Associated to

$$\mathrm{id} \in \mathrm{End}(\mathrm{Ext}^1(\mathrm{V}, \mathcal{O}_{\mathbb{P}^2_k}(-1))) = \mathrm{Ext}^1(\mathrm{V}, \mathrm{C}' \otimes \mathcal{O}_{\mathbb{P}^2_k}(-1))$$

and to

$$\mathrm{id} \in \mathrm{End}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2_k}(1), \mathrm{V})) = \mathrm{Ext}^1(\mathrm{C}'' \otimes \mathcal{O}_{\mathbb{P}^2_k}(1), \mathrm{V})$$

are the two extensions

$$0 \to \mathrm{C}' \otimes \mathcal{O}_{\mathbb{P}^2_k}(-1) \longrightarrow \mathrm{K} \longrightarrow \mathrm{V} \to 0$$

and

$$0 \to \mathrm{V} \longrightarrow \mathrm{Q} \longrightarrow \mathrm{C}'' \otimes \mathcal{O}_{\mathbb{P}^2_k}(1) \to 0.$$

Since $\mathrm{Ext}^2(\mathrm{C}'' \otimes \mathcal{O}_{\mathbb{P}^2_k}(1), \mathrm{C}' \otimes \mathcal{O}_{\mathbb{P}^2_k}(-1)) = 0$, we can complete these two extensions to a monad display and obtain a monad

$$0 \to \mathrm{C}' \otimes \mathcal{O}_{\mathbb{P}^2_k}(-1) \longrightarrow \tilde{\mathrm{V}} \longrightarrow \mathrm{C}'' \otimes \mathcal{O}_{\mathbb{P}^2_k}(1) \to 0$$

with cohomology V.

It follows immediately that the Chern classes of $\tilde{\mathrm{V}}$ vanish. The special choice of our extensions above imply, that

$$\mathrm{H}^0(\mathrm{C}'' \otimes \mathcal{O}_{\mathbb{P}^2_k}) \to \mathrm{H}^1(\mathrm{V}(-1))$$

and

$$\mathrm{H}^1(\mathrm{V}) \to \mathrm{H}^2(\mathrm{C}' \otimes \mathcal{O}_{\mathbb{P}^2_k})(-1)$$

are essentially identities. Chasing through the cohomology diagrams associated to the monad display yields that the cohomology groups of the twists $\tilde{\mathrm{V}}(\ell)$ are isomorphic to the cohomology groups of the trivial sheaf; thus $\tilde{\mathrm{V}}$ is isomorphic to the trivial sheaf $\mathrm{C} \otimes \mathcal{O}_{\mathbb{P}^2_k}$ with C a k-vector space of dimension 2m+r. □

We examine now the moduli space $\mathcal{V}(\mathrm{m}, \mathrm{r})$ in the classical case that S' is just the projective plane $\mathbb{P}^2_k$ with homogeneous coordinates $(\mathrm{x} : \mathrm{y} : \mathrm{z})$. In the notation of Section 1.3, the fixed line $\mathrm{F}' = \mathrm{F}'_\infty$ will be equal to $\mathrm{Z}(\mathrm{z})$, and the point P' will be $(0 : 1 : 0)$. The following theorem is due to J. Hurtubise [18, Section 3.D].

Theorem 2.4 *A closed point* $(\mathrm{V}, \alpha) \in \mathcal{V}(\mathrm{m}, \mathrm{r})$ *is determined by a monad*

$$0 \to \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(-1) \otimes \mathrm{L} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}} \otimes (\mathrm{L} \oplus \mathrm{L} \oplus \mathrm{k}^{\mathrm{r}}) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(1) \otimes \mathrm{L} \to 0,$$

where the following conditions are fulfilled:

(0) L *is a* k*-vector space of dimension* m*;*

(i)

$$\varphi = \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 0 \\ \mathrm{id} \\ 0 \end{pmatrix} \mathrm{y} + \begin{pmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{pmatrix} \mathrm{z};$$

(ii)

$$\psi = \begin{pmatrix} 0 & -\mathrm{id} & 0 \end{pmatrix} \mathrm{x} + \begin{pmatrix} \mathrm{id} & 0 & 0 \end{pmatrix} \mathrm{y} + \begin{pmatrix} \mathrm{b} & -\mathrm{a} & \mathrm{d} \end{pmatrix} \mathrm{z};$$

(iii) $\mathrm{a}, \mathrm{b} \in \mathrm{End}_{\mathrm{k}}(\mathrm{L})$, $\mathrm{c} \in \mathrm{Hom}_{\mathrm{k}}(\mathrm{L}, \mathrm{k}^{\mathrm{r}})$, $\mathrm{d} \in \mathrm{Hom}_{\mathrm{k}}(\mathrm{k}^{\mathrm{r}}, \mathrm{L})$ *with* $\mathrm{ba} - \mathrm{ab} + \mathrm{dc} = 0$*;*

(iv) φ *and* ψ *are fibrewise of full rank;*

(v) $\Delta(\mathrm{V}, \alpha)$ *equals the zero divisor of* $\det(\mathrm{a} - \xi \mathrm{id})$.

In the case that the vector sheaf V has $\mathrm{F}'_0 = \mathrm{Z}(\mathrm{x})$ *as the only jumping line through the point* $\mathrm{P}' = (0:1:0)$*, then (iv) translates to the condition, that*

$$\mathrm{rk} \begin{pmatrix} \mathrm{a} \\ \mathrm{b} + \xi \mathrm{id} \\ \mathrm{c} \end{pmatrix} = \mathrm{rk} \begin{pmatrix} \mathrm{b} + \xi \mathrm{id} & -\mathrm{a} & \mathrm{d} \end{pmatrix} = \mathrm{m}$$

for all $\xi \in \mathrm{k}$*. If the group* $\mathrm{Gl}(\mathrm{L})$ *acts on the space of such monads by*

$$\mathrm{g} \cdot (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) = (\mathrm{gag}^{-1}, \mathrm{gbg}^{-1}, \mathrm{cg}^{-1}, \mathrm{gd}),$$

then the orbits of this action are in bijection with $\mathcal{V}(\mathrm{m}, \mathrm{r})(\mathrm{Spec}\,\mathrm{k})$.

Proof. We consider a monad

$$0 \to \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(-1) \otimes \mathrm{C}' \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}} \otimes \mathrm{C} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^2_{\mathrm{k}}}(1) \otimes \mathrm{C}'' \to 0,$$

with cohomology V, as provided by Theorem 2.3. The fact that V is trivial along F'_∞ is equivalent by [24, Lemma 2.3.4] to the condition that, for any two distinct points P and Q on F'_∞, the composition $\mathrm{D} = \psi(\mathrm{Q})\varphi(\mathrm{P}) = -\psi(\mathrm{P})\varphi(\mathrm{Q})$ is an isomorphism from C' to C''. We choose $\mathrm{P} = (0:1:0)$ and $\mathrm{Q} = (1:0:0)$ and use $\psi(\mathrm{Q})\varphi(\mathrm{P})$ to identify $\mathrm{L} = \mathrm{C}' = \mathrm{C}''$. Moreover, with $\psi\varphi = 0$ we obtain a decomposition

$$\mathrm{C} = \operatorname{im}\varphi(\mathrm{Q}) \oplus \operatorname{im}\varphi(\mathrm{P}) \oplus (\ker\psi(\mathrm{Q}) \cap \ker\psi(\mathrm{P})) = \mathrm{L} \oplus \mathrm{L} \oplus \mathrm{C}_\infty$$

and can write

$$\varphi = \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 0 \\ \mathrm{id} \\ 0 \end{pmatrix} \mathrm{y} + \begin{pmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{pmatrix} \mathrm{z}$$

and

$$\psi = \begin{pmatrix} 0 & -\mathrm{id} & 0 \end{pmatrix} \mathrm{x} + \begin{pmatrix} \mathrm{id} & 0 & 0 \end{pmatrix} \mathrm{y} + \begin{pmatrix} \mathrm{e} & \mathrm{f} & \mathrm{d} \end{pmatrix} \mathrm{z}.$$

The monad condition $\psi\varphi = 0$ implies $\mathrm{e} = \mathrm{b}$, $\mathrm{f} = -\mathrm{a}$ and $\mathrm{ba} - \mathrm{ab} + \mathrm{dc} = 0$. Hereby, $\mathrm{a}, \mathrm{b} \in \mathrm{End}_{\mathrm{k}}(\mathrm{L})$, $\mathrm{c} \in \mathrm{Hom}_{\mathrm{k}}(\mathrm{L}, \mathrm{C}_\infty)$ and $\mathrm{d} \in \mathrm{Hom}_{\mathrm{k}}(\mathrm{C}_\infty, \mathrm{L})$.

As it can be seen immediately from the restriction of the so-reduced monad to $\mathrm{F}'_\infty = \mathrm{Z}(\mathrm{z})$, the framing α corresponds to an isomorphism $\mathrm{C}_\infty \cong \mathrm{k}^{\mathrm{r}}$. The condition (iv) is necessary to have φ and ψ as injective, respectively surjective, bundle morphisms.

We consider now the pencil $\mathrm{N} = \{\mathrm{F}'_\xi = \mathrm{Z}(\mathrm{x}+\xi\mathrm{z})\}$ of lines through P'. Recall that $\sigma : \mathrm{S} \to \mathrm{S}'$ is the blowing up of S' in P' with the corresponding morphism $\pi : \mathrm{S} \to \mathrm{N}$ and a section $\mathrm{N} \hookrightarrow \mathrm{S}$. Δ maps (V, α) to the determinantal divisor associated to $(\sigma^*\mathrm{V})(-\mathrm{N})$. For fixed ξ, V is free along F'_ξ iff

$$\psi(0:1:0)\varphi(-\xi:0:1) = \mathrm{a} - \xi\mathrm{id}$$

is an isomorphism by [24, Lemma 2.3.4]. Together with the Semicontinuity Theorem, this implies (v).

In the case that V has F_0' as the only jumping line in N, we have to check the condition (*iv*) only in points on $F_0' - \{(0:1:0)\}$ and obtain (*iv*) as equivalent to

$$\mathrm{rk}\begin{pmatrix} a \\ b+\xi \mathrm{id} \\ c \end{pmatrix} = \mathrm{rk}\begin{pmatrix} b+\xi \mathrm{id} & -a & d \end{pmatrix} = m$$

for all $\xi \in k$.

It is evident that any monad given by a quadruple (a, b, c, d) fulfilling the conditions *(0)-(iv)* defines a point in $\mathcal{V}(m, r)$. Conversely, a given point in $\mathcal{V}(m, r)$ defines such a quadruple (a, b, c, d) in the form of matrices up to the choice of a basis in L. We obtain the set $\mathcal{V}(m, r)(\mathrm{Spec}\, k)$ as the quotient of the above formulated action of $\mathrm{Gl}(L)$. □

We define $\mathcal{W}(m, r)$ to be the space of tuples (a, b, c, d) in

$$\mathrm{End}(L) \times \mathrm{End}(L) \times \mathrm{Hom}(L, k^r) \times \mathrm{Hom}(k^r, L)$$

fulfilling the conditions

(1) $ba - ab + dc = 0$;

(2) a is nilpotent;

(3)

$$\mathrm{rk}\begin{pmatrix} a \\ b+\xi \mathrm{id} \\ c \end{pmatrix} = \mathrm{rk}\begin{pmatrix} b+\xi \mathrm{id} & -a & d \end{pmatrix} = m$$

for all $\xi \in k$.

Since the condition (2) is equivalent to the condition that $\det(\xi \mathrm{id} + a)$ is equal to $\xi^m \cdot \mathrm{constant}$, $\mathcal{W}(m, r)$ is the space of those monads which determine the elements in $\mathcal{V}(m, r)(\mathrm{Spec}\, k)$ with the only jumping line F_0' through P'; i.e., $\mathcal{W}(m, r)$ can be considered as the preimage of $\mathcal{J}(m, r)(\mathrm{Spec}\, k)$ in the space of monads fulfilling *(0)-(iv)* under the action of $\mathrm{Gl}(L)$.

Theorem 2.5 *The algebraic space* $\mathcal{J}(\mathrm{m},\mathrm{r})$ *is an algebraic variety over* $\mathrm{Spec}\,\mathrm{k}$ *of dimension* $(2\mathrm{r}-1)\mathrm{m}$*. It equals the geometric quotient of the algebraic space* $\mathcal{W}(\mathrm{m},\mathrm{r})$ *by the action of* $\mathrm{Gl}(\mathrm{L})$*.*

Proof. Since we know already that the functor $\mathcal{J}(\mathrm{m},\mathrm{r})$ is represented by a fine moduli space, we may proceed as follows:

Due to the construction of Theorem 2.4, there is obviously a family of monads on $\mathcal{W}(\mathrm{m},\mathrm{r})$ which could be considered as a universal reduced monad and which induces an element in $\mathcal{J}(\mathrm{m},\mathrm{r})(\mathcal{W}(\mathrm{m},\mathrm{r}))$. This element gives a unique morphism from $\mathcal{W}(\mathrm{m},\mathrm{r})$ to $\mathcal{J}(\mathrm{m},\mathrm{r})$, which is surjective by the previous theorem. From this morphism, we can verify the dimension and the irreducibility of $\mathcal{J}(\mathrm{m},\mathrm{r})$.

But as the theorem is quite important, we will give the self-contained proof from [28] in its full beauty.

Step 1. *Integrality*

Let $\mathcal{W}'(\mathrm{m},\mathrm{r})$ be the closed subspace of tuples $(\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d})$ in

$$\mathrm{End}(\mathrm{L}) \times \mathrm{End}(\mathrm{L}) \times \mathrm{Hom}(\mathrm{L},\mathrm{k}^{\mathrm{r}}) \times \mathrm{Hom}(\mathrm{k}^{\mathrm{r}},\mathrm{L})$$

fulfilling the conditions (1) and (2) from above. We get a projection into the nilpotent cone $\mathcal{N} \subset \mathrm{End}(\mathrm{L})$ with irreducible and reduced image. The fibre $\mathcal{W}'_{\mathrm{a}}$ over $\mathrm{a} \in \mathcal{N}$ projects onto a subset P of $\mathrm{Hom}(\mathrm{L},\mathrm{k}^{\mathrm{r}}) \times \mathrm{Hom}(\mathrm{k}^{\mathrm{r}},\mathrm{L})$, and the condition $(\mathrm{c},\mathrm{d}) \in \mathrm{P}$ is expressed by a system of equations $\ell_\gamma(\mathrm{dc}) = 0$, where ℓ_γ are linear forms on $\mathrm{End}(\mathrm{L})$. Therefore, P is a subvariety, since it projects surjectively onto $\mathrm{Hom}(\mathrm{L},\mathrm{k}^{\mathrm{r}})$ and the fibres are linear subspaces. The fibres of $\mathcal{W}'_{\mathrm{a}} \to \mathrm{P}$ are affine subspaces, hence $\mathcal{W}'_{\mathrm{a}}$, and therefore $\mathcal{W}'(\mathrm{m},\mathrm{r})$, is integral. We note that $\mathcal{W}(\mathrm{m},\mathrm{r})$ is an open subvariety of $\mathcal{W}'(\mathrm{m},\mathrm{r})$, since the condition (3) defines an open part.

An easy dimension count at a general $\mathrm{a} \in \mathcal{N}$ gives $\dim \mathcal{W}'_{\mathrm{a}} = \mathrm{m} + \dim \mathrm{P}$ and $\dim \mathrm{P} = 2\mathrm{rm} - \mathrm{m}$ and hence $\dim \mathcal{W}'_{\mathrm{a}} = 2\mathrm{rm}$. We infer $\dim \mathcal{W}'(\mathrm{m},\mathrm{r}) = 2\mathrm{rm} + \mathrm{m}^2$.

Step 2. $\mathcal{W}(\mathrm{m},\mathrm{r}) \to \mathcal{N}$ *is surjective*

Given $\mathrm{a} \in \mathcal{N}$, we split L into a-invariant subspaces $\mathrm{L}_1 \oplus \ldots \oplus \mathrm{L}_{\mathrm{q}}$ of dimensions $\mathrm{m}_1, \ldots, \mathrm{m}_{\mathrm{q}}$, each L_{i} generated as a $\mathrm{k}[\mathrm{a}]-$module by one vector ℓ_{i}. Then choose b by $\mathrm{b}|_{\mathrm{L}_{\mathrm{i}}} = \gamma_{\mathrm{i}}\mathrm{id}$ with pairwise disjoint $\gamma_{\mathrm{i}} \in \mathrm{k}$ such that the commutator $[\mathrm{b},\mathrm{a}]$ vanishes.

We also choose a non-vanishing linear form $\lambda : \mathrm{k}^{\mathrm{r}} \to \mathrm{k}$ and a non-vanishing vector $\mathrm{w} \in \ker(\lambda)$. Since

$$\mathrm{rk}\begin{pmatrix} \mathrm{a} \\ \gamma \mathrm{id} + \mathrm{b} \end{pmatrix} = \mathrm{rk}\begin{pmatrix} \gamma\mathrm{id} + \mathrm{b} & -\mathrm{a} \end{pmatrix} = \begin{cases} \mathrm{m} & \text{if } \gamma \in \{\gamma_{\mathrm{i}}\} \\ \mathrm{m}-1 & \text{if } \gamma \notin \{\gamma_{\mathrm{i}}\}, \end{cases}$$

we can choose

$$\mathrm{d}(\mathrm{v}) = \lambda(\mathrm{v})(\ell_1 + \ldots + \ell_{\mathrm{q}})$$

and

$$\mathrm{c}(\mathrm{a}^{\nu}\ell_{\mathrm{i}}) = \begin{cases} 0 & \text{if } \nu < \mathrm{m}_{\mathrm{i}} - 1 \\ \mathrm{w} & \text{if } \nu = \mathrm{m}_{\mathrm{i}} - 1. \end{cases}$$

Then the condition (1) is fulfilled because $\mathrm{ba} - \mathrm{ab} = \mathrm{dc} = 0$, and the condition (3) is fulfilled as well.

Step 3. *Stability*

We check stability by the Hilbert-Mumford criterion (cf. [34, Theorem 2.1]), so we have to consider 1-parameter subgroups $\mathrm{g} : \mathbb{G}_{\mathrm{m}} \to \mathrm{Gl}(\mathrm{L})$ and to show that $\lim_{\mathrm{t}\to\infty}\mathrm{g}(\mathrm{t})(\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d})$ never exists.

Given $\mathrm{g}(\mathrm{t})$, we can split L into $\mathrm{L}_1 \oplus \ldots \oplus \mathrm{L}_{\mathrm{q}}$ such that $\mathrm{g}(\mathrm{t}) = \mathrm{t}^{\omega_{\mathrm{j}}}$ on L_{j} with $\omega_1 > \ldots > \omega_{\mathrm{q}}$. We first assume $\omega_1 > 0$. We let $1 < \mathrm{s} \leq \mathrm{q}$ be the last index such that ω_{s} is positive, and let $\mathrm{s} < \sigma \leq \mathrm{q}$ be the first index such that ω_σ is negative. According to the decomposition of L, we write our homomorphisms in blocks

$$\mathrm{a} = (\mathrm{a}_{\mathrm{ij}}),\ \mathrm{b} = (\mathrm{b}_{\mathrm{ij}}),\ \mathrm{c} = (\mathrm{c}_1 \ldots \mathrm{c}_{\mathrm{q}}),\ \mathrm{d} = \begin{pmatrix} \mathrm{d}_1 \\ \vdots \\ \mathrm{d}_{\mathrm{q}} \end{pmatrix}.$$

Suppose $\lim_{\mathrm{t}\to\infty}\mathrm{g}(\mathrm{t})(\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d})$ exists. Since the (i,j)-block of $\mathrm{g}(\mathrm{t})\mathrm{a}\mathrm{g}(\mathrm{t})^{-1}$ is $\mathrm{t}^{\omega_{\mathrm{i}}-\omega_{\mathrm{j}}}$, we have $\mathrm{a}_{\mathrm{ij}} = 0$ for $\mathrm{i} > \mathrm{j}$, and similarly $\mathrm{b}_{\mathrm{ij}} = 0$ for $\mathrm{i} > \mathrm{j}$, $\mathrm{c}_{\mathrm{i}} = 0$ for $\mathrm{i} > \sigma$ and $\mathrm{d}_{\mathrm{j}} = 0$ for $\mathrm{j} \leq \mathrm{s}$. This leads to a contradiction for $(\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}) \in \mathcal{W}(\mathrm{m},\mathrm{r})$:

Considering the (s,s)-blocks in the equation (1) gives

$$\mathrm{b}_{\mathrm{ss}}\mathrm{a}_{\mathrm{ss}} - \mathrm{a}_{\mathrm{ss}}\mathrm{b}_{\mathrm{ss}} = \mathrm{d}_{\mathrm{s}}\mathrm{c}_{\mathrm{s}} = 0.$$

On the other side, the rank condition (3) implies

$$\mathrm{rk}(\xi \mathrm{id}_{\mathrm{L_s}} + \mathrm{b_{ss}} \quad \mathrm{a_{ss}}) = \dim \mathrm{L_s}$$

for all $\xi \in \mathrm{k}$. Since $\mathrm{a_{ss}}$ is nilpotent, $\ker \mathrm{a}_{\mathrm{ss}}^{\vee}$ is not trivial. If $\ell \in \ker \mathrm{a}_{\mathrm{ss}}^{\vee}$, then $\ell \mathrm{a_{ss}} \mathrm{b_{ss}} = \ell \mathrm{b_{ss}} \mathrm{a_{ss}} = 0$, hence $\mathrm{b}_{\mathrm{ss}}^{\vee}$ is an endomorphism on $\ker \mathrm{a}_{\mathrm{ss}}^{\vee}$. Therefore, there is a non-vanishing linear form ℓ on $\mathrm{L_s}$ such that $\ell \circ \mathrm{a_{ss}} = 0$ and $\ell \circ \mathrm{b_{ss}} = \beta_0 \ell$ with some constant $\beta_0 \in \mathrm{k}$. With $\xi = -\beta_0$ we obtain

$$\ell \circ (\xi \mathrm{id}_{\mathrm{L_s}} + \mathrm{b_{ss}} \quad \mathrm{a_{ss}}) = 0$$

and hence

$$\mathrm{rk}(\xi \mathrm{id}_{\mathrm{L_s}} + \mathrm{b_{ss}} \quad \mathrm{a_{ss}}) < \dim \mathrm{L_s},$$

which is a contradiction. Obviously, we may deal with the case $\omega_1 \leq 0$ in the same way by interchanging the role of s and σ. This proves the stability of points in $\mathcal{W}(\mathrm{m}, \mathrm{r})$.

Step 4. *The action of* $\mathrm{Gl}(\mathrm{L})$ *on* $\mathcal{W}(\mathrm{m}, \mathrm{r})$ *is free*

By step 3, the orbits of points of $\mathcal{W}(\mathrm{m}, \mathrm{r})$ are closed and the isotropy group is finite. If $\mathrm{g} \neq \mathrm{id}$ is in the isotropy group of $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) \in \mathcal{W}(\mathrm{m}, \mathrm{r})$, we can split $\mathrm{L} = \mathrm{L}_0 \oplus \ldots \oplus \mathrm{L_p}$ into eigenspaces of g with $\mathrm{g}|_{\mathrm{L}_0} = \mathrm{id}_{\mathrm{L}_0}$ and $\mathrm{g}|_{\mathrm{L_i}} = \varepsilon_{\mathrm{i}} \mathrm{id}_{\mathrm{L_i}}$ for $\mathrm{i} = 1, \ldots, \mathrm{p}$, where the ε_{i} are certain roots of unity not equal to 1.

Corresponding to this splitting, we write our homomorphisms as blocks

$$\mathrm{a} = \begin{pmatrix} \mathrm{a}_0 & & 0 \\ & \ddots & \\ 0 & & \mathrm{a_p} \end{pmatrix}, \ \mathrm{b} = \begin{pmatrix} \mathrm{b}_0 & & 0 \\ & \ddots & \\ 0 & & \mathrm{b_p} \end{pmatrix}, \ \mathrm{c} = (\mathrm{c}_0 \ 0 \ldots 0), \ \mathrm{d} = \begin{pmatrix} \mathrm{d}_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where we have applied $\mathrm{gd} = \mathrm{d}$ and $\mathrm{cg}^{-1} = \mathrm{c}$. We obtain $[\mathrm{a}_0, \mathrm{b}_0] = \mathrm{d}_0 \mathrm{c}_0$, $[\mathrm{a_i}, \mathrm{b_i}] = 0$ and $\mathrm{rk}(\xi \mathrm{id}_{\mathrm{L_i}} + \mathrm{b_i} \quad \mathrm{a_i}) = \dim \mathrm{L_i}$ for $\mathrm{i} > 0$ and all $\xi \in \mathrm{k}$, which again is a contradiction.

Step 5. *Existence of a geometric quotient*

The set $\mathcal{W}'_{\mathrm{s}}(\mathrm{m}, \mathrm{r})$ of stable points in $\mathcal{W}'(\mathrm{m}, \mathrm{r})$ admits a geometric quotient

$$\mathcal{W}'_{\mathrm{s}}(\mathrm{m}, \mathrm{r}) \to \mathrm{Gl}(\mathrm{L}) \setminus \mathcal{W}'_{\mathrm{s}}(\mathrm{m}, \mathrm{r}).$$

Since $\mathcal{W}(\mathrm{m},\mathrm{r})$ is an open and $\mathrm{Gl}(\mathrm{L})$-invariant subvariety of $\mathcal{W}'_{\mathrm{s}}(\mathrm{m},\mathrm{r})$, the restriction

$$\mathcal{W}(\mathrm{m},\mathrm{r}) \to \mathrm{Gl}(\mathrm{L}) \setminus \mathcal{W}(\mathrm{m},\mathrm{r})$$

is a geometric quotient of $\mathcal{W}(\mathrm{m},\mathrm{r})$. Since the action is free, the image is again a variety, and the dimension is $\dim \mathcal{W}(\mathrm{m},\mathrm{r}) - \dim \mathrm{Gl}(\mathrm{L}) = (2\mathrm{r}-1)\mathrm{m}$. Because of Theorem 2.4, this variety is just $\mathcal{J}(\mathrm{m},\mathrm{r})$. □

Corollary 2.6 *The subspace of* $\mathcal{J}(\mathrm{m},\mathrm{r})$ *defined by the condition* $\mathrm{ord}_0 = 1$ *is an open and dense subvariety.*

Proof. The condition $\mathrm{ord}_0 = 1$ defines an open subfunctor because of the Semicontinuity Theorem. This subfunctor is non-empty, as the transition function

$$\begin{pmatrix} \mathrm{z}^{\mathrm{m}} & \mathrm{t}+\mathrm{z} & 0 & \cdots & 0 \\ 0 & \mathrm{z}^{-\mathrm{m}} & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix} \in \mathrm{Gl}_{\mathrm{r}}(\mathrm{k}[\mathrm{t}]_{(\mathrm{t})}[\mathrm{z},\mathrm{z}^{-1}])$$

defines an ordinary jump V of rank r on the local line Y with $\mathrm{ord}\,\mathrm{V} = \mathrm{m}$ and $\mathrm{ord}_0\mathrm{V} = 1$. Since $\mathcal{J}(\mathrm{m},\mathrm{r})$ is irreducible by Theorem 2.5, this subfunctor is dense. □

2.3 Monadic description of exceptional jumps

We consider a blowing up $\mathrm{S}' \to \mathbb{P}^2$ of n distinct points on the projective plane with associated exceptional lines $\mathrm{E}_1,\dots,\mathrm{E}_{\mathrm{n}}$. For a fixed point $\mathrm{P} \in \mathrm{S}'$ outside of the exceptional locus we let $\{\mathrm{F}'_\xi \mid \xi \in \mathbb{P}^1_{\mathrm{k}}\}$ be the pencil of the total transforms of lines through P. We assume that F'_∞ is disjoint to the exceptional locus. If F'_ξ is the rational curve in the above pencil containing E_{i}, then we define L'_{i} to be the second component of F'_ξ. We note that L'_{i} is numerically effective and $(\mathrm{E}_{\mathrm{i}} \cdot \mathrm{L}'_{\mathrm{i}}) = 1$. Moreover, there is a short exact sequence

$$0 \to \mathcal{O}_{\mathrm{S}'}(-\mathrm{L}'_{\mathrm{i}}) \longrightarrow \mathcal{O}^2_{\mathrm{S}'} \longrightarrow \mathcal{O}_{\mathrm{S}'}(\mathrm{L}'_{\mathrm{i}}) \to 0.$$

For notational reasons, we set $\mathrm{F}'_\infty = \mathrm{L}'_0$. The following theorem is essentially due to A. King, [24, Theorem 3.3.2].

Theorem 2.7 *Let* V' *be a vector sheaf on* S' *with* $\mathrm{V}'|_{\mathrm{F}'_\infty} \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{F}'_\infty}$ *and with second Chern class* $\mathrm{c}_2\mathrm{V}' = \mathrm{m}$. *There is a monad*

$$0 \to \bigoplus_{\mathrm{i}=0}^{\mathrm{n}} \mathrm{C}'_\mathrm{i} \otimes \mathcal{O}_{\mathrm{S}'}(-\mathrm{L}'_\mathrm{i}) \xrightarrow{\varphi} \mathrm{C} \otimes \mathcal{O}_{\mathrm{S}'} \xrightarrow{\psi} \bigoplus_{\mathrm{i}=0}^{\mathrm{n}} \mathrm{C}''_\mathrm{i} \otimes \mathcal{O}_{\mathrm{S}'}(\mathrm{L}'_\mathrm{i}) \to 0$$

with cohomology V'. *Hereby,* C'_i *and* C''_i *are vector spaces of dimension* m *and* C *is a vector space of dimension* $2\mathrm{m}(\mathrm{n}+1)+\mathrm{r}$.

Our proof of this theorem will be much different from the proof given by King, less natural but considerably shorter. The idea is quite simple: After blowing up with center in P, we construct first a monad in a neighbourhood of all exceptional or jumping fibres using the fact that nice short exact sequences on the union D of these fibres can be lifted without obstruction to suitable sequences creating the display of the desired monad. To obtain a monad on the whole surface, we glue this monad together with a trivial monad on the complement of D. Finally, we just take the direct image of this monad on S', which will be a monad again.

Proof. We consider the blow up $\sigma : \mathrm{S} \to \mathrm{S}'$ with centre in P and exceptional divisor N, and put L_i to be the total transform of L'_i, F_ξ to be the strict transform of F'_ξ and $\mathrm{V} = \sigma^*\mathrm{V}'$. As E_i does not meet P, we denote its pullback again with E_i. We also fix the natural map $\pi : \mathrm{S} \to \mathrm{N}$.

As V is trivial along F_∞, there are only finitely many fibres $\mathrm{F}_0, \ldots, \mathrm{F}_\mathrm{q}$ which are either exceptional or where the restriction of V is not free. We set $\mathrm{D} = \mathrm{F}_0 \cup \ldots \cup \mathrm{F}_\mathrm{q}$. D is a union of smooth rational curves ℓ which either intersect N transversally or equal some E_i.

For ℓ of the first kind, we choose an extension

$$0 \to \mathrm{V}|_\ell \longrightarrow \mathrm{R}'_\ell \longrightarrow \mathrm{C}''_0 \otimes \mathcal{O}_\ell(\mathrm{L}_0) \to 0,$$

and for ℓ of the second kind an extension

$$0 \to \mathrm{V}|_\ell \longrightarrow \mathrm{R}'_\ell \longrightarrow \mathrm{C}''_\mathrm{i} \otimes \mathcal{O}_\ell(\mathrm{L}_\mathrm{i}) \to 0,$$

such that in both cases $\mathrm{H}^1(\ell, \mathrm{R}'_\ell(-1)) = 0$, which is possible since m is equal to $\dim_k \mathrm{C}''_i = c_2\mathrm{V} \geq h^1(\ell, \mathrm{V}|_\ell(-1))$. We define

$$0 \to \mathrm{V}|_\ell \longrightarrow \mathrm{R}_\ell \longrightarrow \bigoplus_{i=0}^{n} \mathrm{C}''_i \otimes \mathcal{O}_\ell(\mathrm{L}_i) \to 0$$

by adding the missing direct summands to R'_ℓ. Due to our definitions, $(\ell \cdot \mathrm{L}_i) \geq 0$ and therefore $\mathrm{H}^1(\ell, \mathrm{R}_\ell(-1)) = 0$ for all appearing components $\ell \subset \mathrm{D}$.

We can glue together these extensions in the intersection points of the components of D to obtain an extension

$$0 \to \mathrm{V}|_\mathrm{D} \longrightarrow \mathrm{R}_\mathrm{D} \longrightarrow \bigoplus_{i=0}^{n} \mathrm{C}''_i \otimes \mathcal{O}_\mathrm{D}(\mathrm{L}_i) \to 0,$$

where the property $\mathrm{H}^1(\ell, \mathrm{R}_\mathrm{D}|_\ell(-1)) = 0$ still holds.

Since the cohomological dimension of $\mathrm{S}_0 = \mathrm{S} - \mathrm{F}_\infty$ is one, there are no obstructions to lifting the last short exact sequence to an extension

$$f_0 = (0 \to \mathrm{V}|_{\mathrm{S}_0} \longrightarrow \mathrm{R}_{\mathrm{S}_0} \longrightarrow \bigoplus_{i=0}^{n} \mathrm{C}''_i \otimes \mathcal{O}_{\mathrm{S}_0}(\mathrm{L}_i) \to 0).$$

In exactly the same manner we can obtain an extension

$$e_\mathrm{D} = (0 \to \bigoplus_{i=0}^{n} \mathrm{C}'_i \otimes \mathcal{O}_\mathrm{D}(-\mathrm{L}_i) \longrightarrow \mathcal{M}_\mathrm{D} \longrightarrow \mathrm{R}_\mathrm{D} \to 0)$$

with the property that $\mathcal{M}_\mathrm{D}|_\ell$ is free for all irreducible components $\ell \subset \mathrm{D}$. We can find $\mathrm{D} \subset \mathrm{S}_3 \subset \mathrm{S}_0$ as a complement of finitely many fibres and a lift

$$e_3 = (0 \to \bigoplus_{i=0}^{n} \mathrm{C}'_i \otimes \mathcal{O}_{\mathrm{S}_3}(-\mathrm{L}_i) \longrightarrow \mathcal{M}_{\mathrm{S}_3} \longrightarrow \mathrm{R}_{\mathrm{S}_0}|_{\mathrm{S}_3} \to 0)$$

of e_D, such that $\mathcal{M}_{\mathrm{S}_3}$ is free on all fibres of $\pi|_{\mathrm{S}_3}$ and therefore free. On the other side, we consider $\mathrm{S}_2 = \mathrm{S}_0 - \mathrm{D} \subset \mathrm{S}_0$ and choose an extension

$$e_2 = (0 \to \bigoplus_{i=0}^{n} \mathrm{C}'_i \otimes \mathcal{O}_{\mathrm{S}_2}(-\mathrm{L}_i) \longrightarrow \mathcal{M}_{\mathrm{S}_2} \longrightarrow \mathrm{R}_{\mathrm{S}_0}|_{\mathrm{S}_2} \to 0)$$

with $\mathcal{M}_{\mathrm{S}_2}$ free. This can be done since $f_0|_{\mathrm{S}_2}$ splits. Obviously, we can glue e_2 and e_3 together to an extension

$$0 \to \bigoplus_{i=0}^{n} \mathrm{C}'_i \otimes \mathcal{O}_{\mathrm{S}_0}(-\mathrm{L}_i) \longrightarrow \mathcal{M}_{\mathrm{S}_0} \longrightarrow \mathrm{R}_{\mathrm{S}_0} \to 0$$

with the property that $\mathcal{M}_{S_0}$ is free.

We define a vector sheaf K_{S_0} on S_0 as the fibre product of the two morphisms $V|_{S_0} \to R_{S_0}$ and $\mathcal{M}_{S_0} \to R_{S_0}$ and obtain a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus_{i=0}^{n} C_i' \otimes \mathcal{O}_{S_0}(-L_i) & \longrightarrow & K_{S_0} & \longrightarrow & V|_{S_0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus_{i=0}^{n} C_i' \otimes \mathcal{O}_{S_0}(-L_i) & \longrightarrow & \mathcal{M}_{S_0} & \longrightarrow & R_{S_0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & \bigoplus_{i=0}^{n} C_i'' \otimes \mathcal{O}_{S_0}(L_i) & \longrightarrow & \bigoplus_{i=0}^{n} C_i'' \otimes \mathcal{O}_{S_0}(L_i) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This diagram is the display of a monad

$$\mathcal{M}_0^\bullet = (0 \to \bigoplus_{i=0}^{n} C_i' \otimes \mathcal{O}_{S_0}(-L_i) \xrightarrow{\varphi_0} \mathcal{M}_{S_0} \xrightarrow{\psi_0} \bigoplus_{i=0}^{n} C_i'' \otimes \mathcal{O}_{S_0}(L_i) \to 0)$$

of vector sheaves on $S_0 = S - F_\infty$ with cohomology $V|_{S_0}$.

Let be $S_1 = S - D$. We choose an extension

$$0 \to \bigoplus_{i=0}^{n} C_i' \otimes \mathcal{O}_{S_1}(-L_i) \longrightarrow \mathcal{O}_{S_1}^{2m(n+1)} \longrightarrow \bigoplus_{i=0}^{n} C_i'' \otimes \mathcal{O}_{S_1}(L_i) \to 0$$

and obtain a monad

$$\mathcal{M}_1^\bullet = (0 \to \bigoplus_{i=0}^{n} C_i' \otimes \mathcal{O}_{S_1}(-L_i) \xrightarrow{\varphi_1} \mathcal{M}_{S_1} \xrightarrow{\psi_1} \bigoplus_{i=0}^{n} C_i'' \otimes \mathcal{O}_{S_1}(L_i) \to 0)$$

by adding $\mathcal{O}_{S_1}^r$ to the middle term.

We denote with $\mathcal{M}_2^\bullet$ the restriction of $\mathcal{M}_1^\bullet$ to S_2. We can find an isomorphism from $\mathcal{M}_0^\bullet|_{S_2}$ to $\mathcal{M}_2^\bullet$ as follows: Since $\mathrm{Ext}^1(\mathcal{O}_{S_2}(L_i), V|_{S_2}) = 0$ for all i, we can fix a splitting

$$R_{S_0}|_{S_2} \cong V|_{S_2} \oplus \bigoplus_{i=0}^{n} C_i'' \otimes \mathcal{O}_{S_2}(L_i).$$

We infer a splitting

$$\mathcal{M}_{S_0}|_{S_2} \cong V|_{S_2} \oplus \mathcal{O}_{S_2}^{2m(n+1)}.$$

Any isomorphism $V|_{S_2} \cong \mathcal{O}_{S_2}^r$ induces now the desired isomorphism of monads.

We denote with j_i the embedding of S_i into S. Due to the above isomorphism we have a morphism $j_{0*}\mathcal{M}_0^\bullet \to j_{2*}\mathcal{M}_2^\bullet$. The natural restriction affords the further morphism $j_{1*}\mathcal{M}_1^\bullet \to j_{2*}\mathcal{M}_2^\bullet$. The product

$$\mathcal{M}^\bullet = j_{0*}\mathcal{M}_0^\bullet \times_{j_{2*}\mathcal{M}_2^\bullet} j_{1*}\mathcal{M}_1^\bullet$$

is evidently a monad of vector sheaves of the form

$$0 \to \bigoplus_{i=0}^{n} C_i' \otimes \mathcal{O}_S(-L_i) \xrightarrow{\varphi} \mathcal{M} \xrightarrow{\psi} \bigoplus_{i=0}^{n} C_i'' \otimes \mathcal{O}_S(L_i) \to 0$$

and has cohomology V by construction.

Also by construction, $\mathcal{M}$ is free along fibres of π.

$R^1\sigma_*V(\pm N) = 0$ and $R^1\sigma_*\mathcal{O}_S(L_i)(\pm N) = 0$ imply $R^1\sigma_*R_S(\pm N) = 0$, where R_S denotes the cokernel of φ. With $R^1\sigma_*\mathcal{O}_S(-L_i)(\pm N) = 0$ we obtain $R^1\sigma_*\mathcal{M}(\pm N) = 0$ and hence that $\mathcal{M}$ is free along N. Therefore, it is free on S and hence isomorphic to $C \otimes \mathcal{O}_S$. $\sigma_*\mathcal{M}^\bullet$ is the desired monad for V'. □

The previous theorem opens a way for the monadic description of spaces $\mathcal{E}(m,r)$ of exceptional framed jumps introduced in Section 2.1. We consider S' as a blowing up of $\mathbb{P}^2 = \operatorname{Proj} k[u_0, u_1, u_2]$ in the point $(0:1:0)$. We denote with H the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ and with E the exceptional divisor on S'. Out of E, we use the homogeneous coordinates of $\mathbb{P}^2$ to describe points on S'. We note, that $H^0(S', H) = \operatorname{span}(u_0, u_1, u_2)$ and $H^0(S', H(-E)) = \operatorname{span}(u_0, u_2)$. We also consider the pencil of rational curves $\{F'_\xi = Z(\xi u_1 + u_2)\}$ through the point $P' = (1:0:0)$, with $F'_\infty = Z(u_1)$.

Proposition 2.8 *The algebraic space* $\mathcal{E}(m,r)$ *is naturally isomorphic to the closed subfunctor of* $\mathcal{V}ec(S', F'_\infty, \mathcal{O}_{F'_\infty}, c_1 = 0, c_2 = m)$ *parametrising all those families* (V, α)*, where* V *is trivial along all* F_ξ *with* $\xi \neq 0$.

Proof. Note, that $\mathcal{V}ec(S', F'_\infty, \mathcal{O}_{F'_\infty}, c_1 = 0, c_2 = m)$ is just our moduli space $\mathcal{I}(m,r)$ from Section 1.2 in the case that S' is the blowing up of the plane in one point. This space is isomorphic to $\mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r})$ by Theorem 1.10, where S is the blowing up of S' in a point on F'_∞. The statement is thus a consequence from the definition of the space $\mathcal{E}(\mathrm{m},\mathrm{r})$ in Section 2.1. □

Theorem 2.9

(I) *A closed point* (V,α) *in* $\mathcal{E}(\mathrm{m},\mathrm{r})$ *is determined by a monad of the form*

$$0\to \mathrm{L}_0\otimes\mathrm{H}^\vee\oplus\mathrm{L}_1\otimes\mathrm{H}^\vee(\mathrm{E})\xrightarrow{\varphi}(\mathrm{L}_0\oplus\mathrm{L}_1\oplus\mathrm{L}_0\oplus\mathrm{L}_1\oplus\mathrm{k}^\mathrm{r})\otimes\mathcal{O}_{\mathrm{S}'}\xrightarrow{\psi}\mathrm{C}''_0\otimes\mathrm{H}\oplus\mathrm{C}''_1\otimes\mathrm{H}(-\mathrm{E})\to 0$$

with

(0) L_0 *and* L_1 *are* k*-vector spaces of dimension* m*;*

(i)

$$\varphi=\begin{pmatrix} \mathrm{id}_{\mathrm{L}_0}\mathrm{u}_0-\mathrm{dau}_1 & 0\\ \mathrm{au}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_0\\ \mathrm{id}_{\mathrm{L}_0}\mathrm{u}_2-\mathrm{dbu}_1 & 0\\ \mathrm{bu}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_2\\ \mathrm{eu}_1 & 0\end{pmatrix};$$

(ii)

$$\psi=\begin{pmatrix} \mathrm{bu}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_2 & -\mathrm{au}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_0 & \mathrm{fu}_1\\ \mathrm{id}_{\mathrm{L}_0}\mathrm{u}_2 & \mathrm{du}_2 & -\mathrm{id}_{\mathrm{L}_0}\mathrm{u}_0 & -\mathrm{du}_0 & 0\end{pmatrix};$$

(iii) $\mathrm{a},\mathrm{b}\in\mathrm{Hom}(\mathrm{L}_0,\mathrm{L}_1),\ \mathrm{d}\in\mathrm{Hom}(\mathrm{L}_1,\mathrm{L}_0),\ \mathrm{e}\in\mathrm{Hom}(\mathrm{L}_0,\mathrm{k}^\mathrm{r}),\ \mathrm{f}\in\mathrm{Hom}(\mathrm{k}^\mathrm{r},\mathrm{L}_1)$ *with* $-\mathrm{bda}+\mathrm{adb}+\mathrm{fe}=0$*;*

(iv) the linear maps

$$\begin{pmatrix}\lambda\mathrm{id}-\mathrm{da}\\ -\mathrm{db}\\ \mathrm{b}\\ \mathrm{e}\end{pmatrix},\ \begin{pmatrix}-\mathrm{da}\\ \mathrm{a}-\lambda\mathrm{b}\\ -\mathrm{db}\\ \mathrm{e}\end{pmatrix},\ \begin{pmatrix}\mathrm{b} & \lambda\mathrm{id}+\mathrm{ad} & \mathrm{f}\end{pmatrix}\ \text{and}\ \begin{pmatrix}-\mathrm{bd} & \lambda\mathrm{b}-\mathrm{a} & \mathrm{f}\end{pmatrix}$$

have full rank m *for all* $\lambda\in\mathrm{k}$*;*

(v) the product bd *is nilpotent.*

(II) *We define* $\mathcal{Z}(\mathrm{m},\mathrm{r})$ *as the locally closed subspace of*

$$\mathrm{Hom}(\mathrm{L}_0,\mathrm{L}_1)\times\mathrm{Hom}(\mathrm{L}_0,\mathrm{L}_1)\times\mathrm{Hom}(\mathrm{L}_1,\mathrm{L}_0)\times\mathrm{Hom}(\mathrm{L}_0,\mathrm{k}^{\mathrm{r}})\times\mathrm{Hom}(\mathrm{k}^{\mathrm{r}},\mathrm{L}_1)$$

of all tuples $(\mathrm{a},\mathrm{b},\mathrm{d},\mathrm{e},\mathrm{f})$ *which fulfill the conditions (iii), (iv) and (v). If the group* $\mathrm{Gl}(\mathrm{L}_0)\times\mathrm{Gl}(\mathrm{L}_1)$ *acts on* $\mathcal{Z}(\mathrm{m},\mathrm{r})(\mathrm{Spec}\,\mathrm{k})$ *by*

$$(\mathrm{g}_0,\mathrm{g}_1)(\mathrm{a},\mathrm{b},\mathrm{e},\mathrm{f},\mathrm{d})=(\mathrm{g}_1\mathrm{a}\mathrm{g}_0^{-1},\mathrm{g}_1\mathrm{b}\mathrm{g}_0^{-1},\mathrm{e}\mathrm{g}_0^{-1},\mathrm{g}_1\mathrm{f},\mathrm{g}_0\mathrm{d}\mathrm{g}_1^{-1}),$$

then $\mathcal{E}(\mathrm{m},\mathrm{r})(\mathrm{Spec}\,\mathrm{k})$ *is in natural bijection to the set of orbits of this action.*

Proof. Due to the previous proposition, we may consider instead of our closed point in $\mathcal{E}(\mathrm{m},\mathrm{r})$ an element (V,α) in $\mathcal{V}ec(\mathrm{S}',\mathrm{F}'_\infty,\mathcal{O}_{\mathrm{F}'_\infty},\mathrm{c}_1=0,\mathrm{c}_2=\mathrm{m})(\mathrm{Spec}\,\mathrm{k})$ with the property that the only jumping line of V through $(1:0:0)$ is the exceptional line F'_0. For this framed vector sheaf, we reproduce the reduction of the monad given by Theorem 2.7 analogously to [24], Section 3.4.

We may write

$$\varphi=\begin{pmatrix}\varphi_{00}\mathrm{u}_0+\varphi_{01}\mathrm{u}_1+\varphi_{02}\mathrm{u}_2 & \varphi_{10}\mathrm{u}_0+\varphi_{12}\mathrm{u}_2\end{pmatrix},$$
$$\psi=\begin{pmatrix}\psi_{00}\mathrm{u}_0+\psi_{01}\mathrm{u}_1+\psi_{02}\mathrm{u}_2\\ \psi_{10}\mathrm{u}_0+\psi_{12}\mathrm{u}_2\end{pmatrix},$$

where $\varphi_{\mathrm{ij}}\in\mathrm{Hom}(\mathrm{C}'_{\mathrm{i}},\mathrm{L})$ and $\psi_{\mathrm{ij}}\in\mathrm{Hom}(\mathrm{L},\mathrm{C}''_{\mathrm{i}})$.

The fact, that V is trivial along F'_∞ implies by [24, Lemma 2.2.4], that for any two distinct points P and Q on F'_∞ the composition $\mathrm{D}=\psi(\mathrm{Q})\varphi(\mathrm{P})=-\psi(\mathrm{P})\varphi(\mathrm{Q})$ is an isomorphism $\mathrm{C}'_0\oplus\mathrm{C}'_1\cong\mathrm{C}''_0\oplus\mathrm{C}''_1$. We choose $\mathrm{P}=(1:0:0)$ and $\mathrm{Q}=(0:0:1)$ and obtain

$$\mathrm{D}=\begin{pmatrix}\mathrm{D}_{00}&\mathrm{D}_{01}\\ \mathrm{D}_{10}&\mathrm{D}_{11}\end{pmatrix}=\begin{pmatrix}\psi_{02}\varphi_{00}&\psi_{02}\varphi_{10}\\ \psi_{12}\varphi_{00}&\psi_{12}\varphi_{10}\end{pmatrix}=-\begin{pmatrix}\psi_{00}\varphi_{02}&\psi_{00}\varphi_{12}\\ \psi_{10}\varphi_{02}&\psi_{10}\varphi_{12}\end{pmatrix}.$$

The automorphism group of $\mathrm{C}'_0\otimes\mathrm{H}^\vee\oplus\mathrm{C}'_1\otimes\mathrm{H}^\vee(\mathrm{E})$ is

$$\begin{pmatrix}\mathrm{Gl}(\mathrm{C}'_0)&0\\ \mathrm{Hom}(\mathrm{C}'_0,\mathrm{C}'_1)&\mathrm{Gl}(\mathrm{C}'_1)\end{pmatrix},$$

and since D is an isomorphism we can replace C_0' by a new C_0', which is in the complement of C_1' as well as in the complement of $\mathrm{D}^{-1}\mathrm{C}_0''$. Then we replace C_1'' by $\mathrm{D}(\mathrm{C}_0')$ and obtain

$$\mathrm{D} = \begin{pmatrix} 0 & \mathrm{D}_{01} \\ \mathrm{D}_{10} & \mathrm{D}_{11} \end{pmatrix}.$$

We have a decomposition $\mathrm{C} = \mathrm{C}(\mathrm{P}) \oplus \mathrm{C}(\mathrm{Q}) \oplus \mathrm{C}_\infty$ with

$$\mathrm{C}(\mathrm{P}) = \operatorname{im}\varphi(\mathrm{P}),\ \mathrm{C}(\mathrm{Q}) = \operatorname{im}\varphi(\mathrm{Q}) \text{ and } \mathrm{C}_\infty = \ker\psi(\mathrm{P}) \cap \ker\psi(\mathrm{Q}).$$

With $\psi_{02}\varphi_{00} = 0$ and $\psi_{02}\varphi_{10}$ an isomorphism we infer

$$\mathrm{C}(\mathrm{P}) = \operatorname{im}\varphi_{00} + \operatorname{im}\varphi_{10} = \operatorname{im}\varphi_{00} \oplus \operatorname{im}\varphi_{10}.$$

Analogously, we obtain

$$\mathrm{C}(\mathrm{Q}) = \operatorname{im}\varphi_{02} + \operatorname{im}\varphi_{12} = \operatorname{im}\varphi_{02} \oplus \operatorname{im}\varphi_{12}.$$

Thus, we may identify

$$\mathrm{L}_0 = \mathrm{C}_0' = \operatorname{im}\varphi_{00} = \operatorname{im}\varphi_{02} = \mathrm{C}_1''$$

and

$$\mathrm{L}_1 = \mathrm{C}_1' = \operatorname{im}\varphi_{10} = \operatorname{im}\varphi_{12} = \mathrm{C}_0''$$

and obtain a decomposition

$$\mathrm{C} = \mathrm{L}_0 \oplus \mathrm{L}_1 \oplus \mathrm{L}_0 \oplus \mathrm{L}_1 \oplus \mathrm{C}_\infty.$$

Moreover, we have

$$\varphi_{00} = \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \varphi_{02} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{id} \\ 0 \\ 0 \end{pmatrix} \quad \varphi_{10} = \begin{pmatrix} 0 \\ \mathrm{id} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \varphi_{12} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{id} \\ 0 \end{pmatrix}$$

and

$$
\begin{aligned}
\psi_{00} &= \begin{pmatrix} 0 & 0 & 0 & -\mathrm{id} & 0 \end{pmatrix}\\
\psi_{02} &= \begin{pmatrix} 0 & \mathrm{id} & 0 & 0 & 0 \end{pmatrix}\\
\psi_{10} &= \begin{pmatrix} 0 & 0 & -\mathrm{id} & -\mathrm{d} & 0 \end{pmatrix}\\
\psi_{12} &= \begin{pmatrix} \mathrm{id} & \mathrm{d} & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

where we have put $\mathrm{d} = \mathrm{D}_{11} \in \mathrm{Hom}(\mathrm{L}_1, \mathrm{L}_0)$. It remains to determine

$$
\varphi_{01} = \begin{pmatrix} \mathrm{a}' \\ \mathrm{a} \\ \mathrm{b}' \\ \mathrm{b} \\ \mathrm{e} \end{pmatrix} \quad \text{and} \quad \psi_{01} = \begin{pmatrix} \mathrm{a}''' & \mathrm{a}'' & \mathrm{b}''' & \mathrm{b}'' & \mathrm{f} \end{pmatrix}.
$$

From the vanishing of $\psi\varphi$ we obtain the conditions

$$
\begin{aligned}
\mathrm{a}'''\mathrm{a}' + \mathrm{a}''\mathrm{a} + \mathrm{b}'''\mathrm{b}' + \mathrm{b}''\mathrm{b} + \mathrm{fe} = 0, &\quad \mathrm{a}''' - \mathrm{b} = 0,\\
\mathrm{b}''' + \mathrm{a} = 0, &\quad \mathrm{a}'' = 0,\\
\mathrm{b}'' = 0, &\quad \mathrm{b}' + \mathrm{db} = 0,\\
\mathrm{a}' + \mathrm{da} = 0, &
\end{aligned}
$$

and conclude

$$
\varphi = \begin{pmatrix} \mathrm{id}_{\mathrm{L}_0}\mathrm{u}_0 - \mathrm{dau}_1 & 0 \\ \mathrm{au}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_0 \\ \mathrm{id}_{\mathrm{L}_0}\mathrm{u}_2 - \mathrm{dbu}_1 & 0 \\ \mathrm{bu}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_2 \\ \mathrm{eu}_1 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \mathrm{bu}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_2 & -\mathrm{au}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_0 & \mathrm{fu}_1 \\ \mathrm{id}_{\mathrm{L}_0}\mathrm{u}_2 & \mathrm{du}_2 & -\mathrm{id}_{\mathrm{L}_0}\mathrm{u}_0 & -\mathrm{du}_0 & 0 \end{pmatrix},
$$

where $\mathrm{a}, \mathrm{b} \in \mathrm{Hom}(\mathrm{L}_0, \mathrm{L}_1)$, $\mathrm{d} \in \mathrm{Hom}(\mathrm{L}_1, \mathrm{L}_0)$, $\mathrm{e} \in \mathrm{Hom}(\mathrm{L}_0, \mathrm{C}_\infty)$, $\mathrm{f} \in \mathrm{Hom}(\mathrm{C}_\infty, \mathrm{L}_1)$, and the condition

$$
-\mathrm{bda} + \mathrm{adb} + \mathrm{fe} = 0
$$

holds in order to guarantee $\psi \circ \varphi = 0$.

The cohomology of the restriction of the monad to F'_∞ is obviously $\mathrm{C}_\infty \otimes \mathcal{O}_{\mathrm{F}'_\infty}$. Hence, the framing $\alpha : \mathrm{V}|_{\mathrm{F}'_\infty} \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{F}'_\infty}$ is fixed by an isomorphism $\mathrm{C}_\infty \cong \mathrm{k}^{\mathrm{r}}$.

Now, we want to implement the property that the only jumping line of V through the point $(1:0:0)$ is the exceptional line $\mathrm{F}'_0 = \mathrm{Z}(\mathrm{u}_2)$. This means that, for all lines $\mathrm{F}'_\xi = \mathrm{Z}(\xi \mathrm{u}_1 + \mathrm{u}_2 = 0)$ with $\xi \neq 0$, we have to guarantee that $\mathrm{V}|_{\mathrm{F}'_\xi}$ is free. By [24, Lemma 2.3.4], this is equivalent to the condition

$$\det(\psi(0:1:-\xi) \circ \varphi(1:0:0)) = |\mathrm{D}_\xi| = \begin{vmatrix} \mathrm{b} & -\xi \mathrm{id} \\ \xi \mathrm{id} & -\xi \mathrm{d} \end{vmatrix} = \xi^* \cdot \mathrm{constant}.$$

We obtain

$$(-\xi)^{-2\mathrm{m}} |\mathrm{D}_\beta| = \begin{vmatrix} -\xi^{-1}\mathrm{b} & \mathrm{id} \\ \mathrm{id} & \mathrm{d} \end{vmatrix} = \begin{vmatrix} 0 & \mathrm{id} + \xi^{-1}\mathrm{bd} \\ \mathrm{id} & \mathrm{d} \end{vmatrix} = (-\xi^{-1})^{\mathrm{m}} |\xi \mathrm{id} - \mathrm{bd}|,$$

where the equality in the middle is an easy application of Gauss elimination. Therefore $\det \mathrm{D}_\xi = \xi^* \cdot \mathrm{constant}$ iff the characteristic polynomial of bd equals ξ^{m}; i.e., iff bd is nilpotent.

To have φ as injective and ψ as surjective bundle morphisms, we have to add the non-degeneracy condition, that both homomorphisms have fibrewise full rank m. So far, this is already clear for points on lines F'_ξ with $\xi \neq 0$. We restrict now φ and ψ first to the strict transform of $\mathrm{Z}(\mathrm{u}_2) - \{(1:0:0)\}$ and then to the exceptional divisor and obtain immediately condition (iv).

We have shown the first claim **(I)** of the theorem, namely the existence of a monad with the properties *(0) - (v)*, that determines a given (V, α). Moreover it is clear that any such reduced monad in $\mathcal{Z}(\mathrm{m}, \mathrm{r})$ determines an element (V, α) in $\mathcal{V}ec(\mathrm{S}', \mathrm{F}'_\infty, \mathcal{O}_{\mathrm{F}'_\infty}, \mathrm{c}_1 = 0, \mathrm{c}_2 = \mathrm{m})(\mathrm{Spec}\,\mathrm{k})$ with the property that the only jumping line of V through $(1:0:0)$ is the exceptional line F'_0; i.e., an element in $\mathcal{E}(\mathrm{m}, \mathrm{r})(\mathrm{Spec}\,\mathrm{k})$. As the spaces L_0 and L_1 are fixed up to isomorphisms, the action of their automorphism groups and thus the induced action of $\mathrm{Gl}(\mathrm{L}_0) \times \mathrm{Gl}(\mathrm{L}_1)$ on $\mathcal{Z}(\mathrm{m}, \mathrm{r})$

$$(\mathrm{g}_0, \mathrm{g}_1)(\mathrm{a}, \mathrm{b}, \mathrm{e}, \mathrm{f}, \mathrm{d}) = (\mathrm{g}_1 \mathrm{a} \mathrm{g}_0^{-1}, \mathrm{g}_1 \mathrm{b} \mathrm{g}_0^{-1}, \mathrm{e} \mathrm{g}_0^{-1}, \mathrm{g}_1 \mathrm{f}, \mathrm{g}_0 \mathrm{d} \mathrm{g}_1^{-1})$$

does not change the determined (V, α). On the other side, it is clear from the above construction, that no other choices can be made, and we obtain

$$\mathcal{E}(\mathrm{m}, \mathrm{r})(\mathrm{Spec}\,\mathrm{k}) = \mathcal{Z}(\mathrm{m}, \mathrm{r})(\mathrm{Spec}\,\mathrm{k}) / \mathrm{Gl}(\mathrm{L}_0) \times \mathrm{Gl}(\mathrm{L}_1). \square$$

Of course, we do not want $\mathcal{E}(\mathrm{m},\mathrm{r})$ just as a set-theoretical quotient but as an algebraic quotient together with some insight in its geometry, which is provided by the next theorem.

Theorem 2.10

(i) The algebraic space $\mathcal{E}(\mathrm{m},\mathrm{r})$ is a reduced algebraic scheme over $(\mathrm{Spec}\,\mathrm{k})$ of dimension $(2\mathrm{r}-1)\mathrm{m}$, consisting of two integral components appearing as the closures of $\sigma_0^*\mathcal{J}(\mathrm{m},\mathrm{r})$ and $\sigma_1^*\mathcal{J}(\mathrm{m},\mathrm{r})$, respectively.

(ii) The action of the algebraic group $\mathrm{Gl}(\mathrm{L}_0)\times\mathrm{Gl}(\mathrm{L}_1)$ on the space $\mathcal{Z}(\mathrm{m},\mathrm{r})$ is free, and there exists a geometric quotient which is equal to $\mathcal{E}(\mathrm{m},\mathrm{r})$.

Proof.

Step 1. The natural morphism $\mathcal{Z}(\mathrm{m},\mathrm{r})\to\mathcal{E}(\mathrm{m},\mathrm{r})$

Due to the construction of Theorem 2.9, there is a monad on $\mathcal{Z}(\mathrm{m},\mathrm{r})\times\mathrm{S}'$, which could be considered as the universal reduced monad. The cohomology of this monad is clearly an element in $\mathcal{E}(\mathrm{m},\mathrm{r})(\mathcal{Z}(\mathrm{m},\mathrm{r}))$. Because of the universal property of a fine moduli space, this element gives rise to a natural morphism $\mathcal{Z}(\mathrm{m},\mathrm{r})\to\mathcal{E}(\mathrm{m},\mathrm{r})$. This morphism is set-theoretically the quotient of the action of $\mathrm{Gl}(\mathrm{L}_0)\times\mathrm{Gl}(\mathrm{L}_1)$ on $\mathcal{Z}(\mathrm{m},\mathrm{r})$ due to Theorem 2.9.(II).

Step 2. The preimage of $\sigma_0^*\mathcal{J}(\mathrm{m},\mathrm{r})$ in $\mathcal{Z}(\mathrm{m},\mathrm{r})$

When does a tuple $(\mathrm{a},\mathrm{b},\mathrm{d},\mathrm{e},\mathrm{f})$ in $\mathcal{Z}(\mathrm{m},\mathrm{r})(\mathrm{Spec}\,\mathrm{k})$ define a monad whose cohomology is trivial along E? We restrict the monad to E and obtain

$$0\to\mathrm{L}_0\otimes\mathcal{O}_\mathrm{E}\oplus\mathrm{L}_1\otimes\mathcal{O}_\mathrm{E}(-1)\xrightarrow{\varphi_\mathrm{E}}(\mathrm{L}_0\oplus\mathrm{L}_1\oplus\mathrm{L}_0\oplus\mathrm{L}_1\oplus\mathrm{k}^\mathrm{r})\otimes\mathcal{O}_\mathrm{E}\xrightarrow{\psi_\mathrm{E}}\mathrm{L}_0\otimes\mathcal{O}_\mathrm{E}\oplus\mathrm{L}_1\otimes\mathcal{O}_\mathrm{E}(1)\to 0$$

with

$$\varphi_\mathrm{E}=\begin{pmatrix} -\mathrm{da} & 0 \\ \mathrm{a} & \mathrm{id}\alpha \\ -\mathrm{db} & 0 \\ \mathrm{b} & \mathrm{id}\beta \\ \mathrm{e} & 0 \end{pmatrix} \text{ and } \psi_\mathrm{E}=\begin{pmatrix} \mathrm{b} & 0 & -\mathrm{a} & 0 & \mathrm{f} \\ \mathrm{id}\beta & \mathrm{d}\beta & -\mathrm{id}\alpha & -\mathrm{d}\alpha & 0 \end{pmatrix},$$

where α and β are the homogeneous coordinates on E obtained by the equation $\frac{u_0}{u_1}\beta - \frac{u_2}{u_1}\alpha$. Since kernel and cokernel of morphisms between trivial bundles are again trivial, we see that the cohomology of the above monad is trivial if the cohomology of the monad

$$0 \to L_1 \otimes \mathcal{O}_E(-1) \xrightarrow{\varphi'_E} (L_0 \oplus L_1 \oplus L_0 \oplus L_1 \oplus k^r) \otimes \mathcal{O}_E \xrightarrow{\psi'_E} L_1 \otimes \mathcal{O}_E(1) \to 0$$

is trivial, where

$$\varphi'_E = \begin{pmatrix} 0 \\ \mathrm{id}\alpha \\ 0 \\ \mathrm{id}\beta \\ 0 \end{pmatrix} \text{ and } \psi'_E = \begin{pmatrix} \mathrm{id}\beta & d\beta & -\mathrm{id}\alpha & -d\alpha & 0 \end{pmatrix}.$$

The criterion of [24, Lemma 2.3.4] tells us that this is exactly the case iff the composition $\psi'_E(1:0) \circ \varphi'_E(0:1) = -d$ is an isomorphism. Hence, the preimage of $\sigma_0^* \mathcal{J}(m, r)$ in $\mathcal{Z}(m, r)$ is given by the condition that d is regular.

Step 3. The preimage of $\sigma_1^* \mathcal{J}(m, r)$ *in* $\mathcal{Z}(m, r)$

In analogy to Step 2, we show when a tuple (a, b, d, e, f) in $\mathcal{Z}(m, r)(\operatorname{Spec} k)$ defines a monad whose cohomology is trivial along the strict transform L of $Z(u_2)$. We restrict the monad to L and obtain

$$0 \to L_0 \otimes \mathcal{O}_L(-1) \oplus L_1 \otimes \mathcal{O}_L \xrightarrow{\varphi_L} (L_0 \oplus L_1 \oplus L_0 \oplus L_1 \oplus k^r) \otimes \mathcal{O}_L \xrightarrow{\psi_L} L_0 \otimes \mathcal{O}_L(1) \oplus L_1 \otimes \mathcal{O}_L \to 0$$

with

$$\varphi_L = \begin{pmatrix} \mathrm{id}_{L_0} u_0 - dau_1 & 0 \\ au_1 & \mathrm{id}_{L_1} \\ -dbu_1 & 0 \\ bu_1 & 0 \\ eu_1 & 0 \end{pmatrix} \text{ and } \psi_L = \begin{pmatrix} bu_1 & 0 & -au_1 & \mathrm{id}_{L_1} u_0 & fu_1 \\ 0 & 0 & -\mathrm{id}_{L_0} & -d & 0 \end{pmatrix}.$$

Since the kernel and cokernel of morphisms between trivial bundles are again trivial, we see that the cohomology of the above monad is trivial if the cohomology of the monad

$$0 \to \mathrm{L}_0 \otimes \mathcal{O}_{\mathrm{L}}(-1) \xrightarrow{\varphi'_{\mathrm{L}}} (\mathrm{L}_0 \oplus \mathrm{L}_1 \oplus \mathrm{L}_0 \oplus \mathrm{L}_1 \oplus \mathrm{k}^{\mathrm{r}}) \otimes \mathcal{O}_{\mathrm{L}} \xrightarrow{\psi'_{\mathrm{L}}} \mathrm{L}_0 \otimes \mathcal{O}_{\mathrm{L}}(1) \to 0$$

is trivial, where

$$\varphi'_{\mathrm{L}} = \begin{pmatrix} \mathrm{id}_{\mathrm{L}_0}\mathrm{u}_0 - \mathrm{dau}_1 \\ \mathrm{au}_1 \\ -\mathrm{dbu}_1 \\ \mathrm{bu}_1 \\ \mathrm{eu}_1 \end{pmatrix} \text{ and } \psi'_{\mathrm{L}} = \begin{pmatrix} \mathrm{bu}_1 & 0 & -\mathrm{au}_1 & \mathrm{id}_{\mathrm{L}_1}\mathrm{u}_0 & \mathrm{fu}_1 \end{pmatrix}.$$

The criterion of [24, Lemma 2.3.4] tells us that this is exactly the case iff the composition $\psi'_{\mathrm{L}}(0:1) \circ \varphi'_{\mathrm{L}}(1:0) = \mathrm{b}$ is an isomorphism. Hence, the preimage of $\sigma_1^*\mathcal{J}(\mathrm{m},\mathrm{r})$ in $\mathcal{Z}(\mathrm{m},\mathrm{r})$ is given by the condition that b is regular.

Step 4. $\sigma_0^*\mathcal{J}(\mathrm{m},\mathrm{r}) \cup \sigma_1^*\mathcal{J}(\mathrm{m},\mathrm{r})$ *is dense in* $\mathcal{E}(\mathrm{m},\mathrm{r})$

Of course, we show that the preimage, given by the condition that b or d is regular, is dense in $\mathcal{Z}(\mathrm{m},\mathrm{r})$.

First, we note that if $\mathrm{b} \in \mathrm{Hom}(\mathrm{L}_0,\mathrm{L}_1)$ and $\mathrm{d} \in \mathrm{Hom}(\mathrm{L}_1,\mathrm{L}_0)$ are both singular with bd nilpotent, then any open neighbourhood of the pair (b,d) in the space $\mathrm{Hom}(\mathrm{L}_0,\mathrm{L}_1) \times \mathrm{Hom}(\mathrm{L}_1,\mathrm{L}_0)$ contains pairs $(\mathrm{b}',\mathrm{d}')$ such that $\mathrm{b}'\mathrm{d}'$ is nilpotent and either b' or d' is regular. Indeed, as we may choose bases of L_0 and L_1 such that bd is in Jordan normal form and

$$\mathrm{b} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix},$$

this is provided by for instance by $\mathrm{b}' = \mathrm{b}$ and

$$\mathrm{d}' = \mathrm{d} + \lambda \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and the observation that d' is regular for general $\lambda \in \mathrm{k}$.

Now we may choose other bases such that d' is the identity and b' is nilpotent in Jordan normal form. From here it is evident that in every neighbourhood of (b,d)

there is a pair (b', d') such that d' is regular, b' is nilpotent and $b'd'$ and $d'b'$ are both nilpotent and of rank $m-1$. The benefit of such a choice is that the linear endomorphism on $\mathrm{Hom}(L_0, L_1)$ which sends a homomorphism a to $ad'b' - b'd'a$ is of corank m. As a side remark, we note that the roles of b and d can be interchanged.

We consider now a point (a, b, d, e, f) in $\mathcal{Z}(m, r)$, with b and d both singular, together with an arbitrary open neighbourhood, which we may assume to be of the form $U \times V \times W$, where

$$a \in U \subset \mathrm{Hom}(L_0, L_1),$$

$$(b, d) \in V \subset \mathrm{Hom}(L_0, L_1) \times \mathrm{Hom}(L_1, L_0) \text{ and}$$

$$(e, f) \in V \subset \mathrm{Hom}(L_0, k^r) \times \mathrm{Hom}(k^r, L_1).$$

We choose $(b', d') \in V$ of the above form. The map from U to $\mathrm{Hom}(L_0, L_1)$ which sends a homomorphism a' to $a'd'b' - b'd'a'$ is hence of corank m, and, for dimensional reasons, we find $a' \in U$ and $(e', f') \in W$ such that $a'd'b' - b'd'a' = -f'e'$. This shows that the condition that either b or d is regular, is dense in $\mathcal{Z}(m, r)$ and consequently, that $\sigma_0^* \mathcal{J}(m, r) \cup \sigma_1^* \mathcal{J}(m, r)$ is dense in $\mathcal{E}(m, r)$.

The steps 1-4 already show the assertion *(i)*. To prove that $\mathcal{E}(m, r)$ is the quotient claimed in *(ii)*, we examine the group action closer.

Step 5. The action of $\mathrm{Gl}(L_0) \times \mathrm{Gl}(L_1)$ *on* $\mathcal{Z}(m, r)$ *is stable*

We check the stability by the Hilbert-Mumford criterion (cf. [34], Theorem 2.1). We consider one parameter subgroups $g = (g_0, g_1) : \mathbb{G}_m \to \mathrm{Gl}(L_0) \times \mathrm{Gl}(L_1)$ and have to show that, for all (a', b', d', e', f') in $\mathcal{Z}(m, r)$, the limit $\lim_{t\to\infty} g(t)(a', b', d', e', f')$ does not exist in $\mathcal{Z}(m, r)$. We assume

$$\lim_{t\to\infty} g(t)(a', b', d', e', f') = (a, b, d, e, f)$$

for some fixed g and (a', b', d', e', f'). We obtain decompositions

$$L_0 = L_{01} \oplus \ldots L_{0s} \oplus \ldots L_{0q}$$
$$L_1 = L_{11} \oplus \ldots L_{1s} \oplus \ldots L_{1p}$$

with the following properties:

- $g_i(t)|_{L_{ij}} = t^{\omega_{ij}} \mathrm{id}_{L_{ij}}$ with $\omega_{ij} \neq \omega_{il}$ for $j \neq l$;
- for $i = 1, \ldots, s$, we have $\omega_{0i} = \omega_{1i} = \omega_i$;
- for all $i, j > s$, we have $\omega_{0i} \neq \omega_{1j}$.

According to this decomposition, we may write a' in blocks $a'_{ij} : L_{0j} \to L_{1i}$. $g(t)$ acts on a'_{ij} by multiplication with $t^{\omega_{1i}-\omega_{0j}}$. If this difference $\omega_{1i} - \omega_{0j}$ is positive, then $a'_{ij} = 0$ is necessary to allow the existence of a limit for $t \to \infty$, and consequently $a_{ij} = 0$. If this difference is negative, then $\lim_{t\to\infty} g(t)(a'_{ij}) = 0$, and hence again $a_{ij} = 0$. Therefore, a is of the form

$$\begin{pmatrix} a_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & a_s & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

with $a_i = a_{ii}$ the only not necessarily vanishing blocks, $i = 1, \ldots, s$. We may treat b and d in a similar fashion to obtain

$$b = \begin{pmatrix} b_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & b_s & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad d = \begin{pmatrix} d_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & d_s & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Moreover, we can write

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} \quad e = \begin{pmatrix} e_1 & \cdots & e_q \end{pmatrix}$$

with the property that $f_i \neq 0$ implies $\omega_{1i} = 0$ and $e_j \neq 0$ implies $\omega_{0j} = 0$.

Now we assume that $\omega_{1i} \neq 0$ for $i > s$. We infer $f_i = 0$, $b_{ij} = 0$ and $(ad)_{ij} = 0$ for all j. This contradicts

$$\mathrm{rk}(b \quad ad \quad f) = m.$$

Therefore, either $p = s + 1$ and $\omega_{1p} = 0$, or $p = s$.

If we assume that $\omega_{0j} \neq 0$ for $j > s$, then we have $e_j = 0$, $(da)_{ij} = (db)_{ij} = 0$ and $b_{ij} = 0$ for all i and obtain a contradiction to

$$\mathrm{rk}\begin{pmatrix} da \\ db \\ b \\ e \end{pmatrix} = m.$$

Hence, either $q = s + 1$ and $\omega_{0q} = 0$, or $q = s$.

Now let us assume that $\omega_i \neq 0$ for some $i \in \{1, \dots, s\}$. We obtain $f_i = 0$, $e_i = 0$ and $a_i d_i b_i = b_i d_i a_i$. We consider three different cases:

Case 1. b_i is not surjective. Because $a_i d_i b_i = b_i d_i a_i$, $(a_i d_i)^\vee$ restricts to an endomorphism on $\ker b_i^\vee$, which is a non-trivial vector space since b_i is not surjective. Therefore, there is a non-vanishing eigenvector $\ell \in \ker b_i^\vee$ with some eigenvalue λ_0, which yields $\ell \circ (b_i \ \lambda_0 \mathrm{id}_{L_{1i}} - a_i d_i) = 0$. Thus, we have a contradiction to

$$\mathrm{rk}(b \quad \lambda_0 \mathrm{id}_{L_1} - ad \quad f) = m.$$

Case 2. b_i is an isomorphism. We may use b_i to identify $L_{0i} = L_{1i}$. We thus have $a_i d_i = d_i a_i$; hence, $a_i^\vee$ maps $\ker d_i^\vee$ to itself. Since $b_i d_i$ is nilpotent, $\ker d_i^\vee$ is not trivial, and $a_i^\vee|_{\ker d_i^\vee}$ admits a non-vanishing eigenvector with some eigenvalue λ_0. This contradicts

$$\mathrm{rk}(bd \quad \lambda_0 b - a \quad f) = m.$$

Case 3. b_i is not injective. With $a_i d_i b_i = b_i d_i a_i$, $d_i a_i$ maps $\ker b_i$ to itself. We choose an eigenvalue λ_0 and obtain again a contradiction, this time to the condition

$$\mathrm{rk}\begin{pmatrix} da \\ db \\ b \\ e \end{pmatrix} = m.$$

We summarize that all ω_{ij} are equal to 0, hence $g(t)$ is constant, and the Step 5 is complete.

Step 6. The action of $\mathrm{Gl}(\mathrm{L}_0) \times \mathrm{Gl}(\mathrm{L}_1)$ *on* $\mathcal{Z}(\mathrm{m}, \mathrm{r})$ *is free*

By Step 5 we already know that all the isotropy groups of the action of the group $\mathrm{Gl}(\mathrm{L}_0) \times \mathrm{Gl}(\mathrm{L}_1)$ on $\mathcal{Z}(\mathrm{m}, \mathrm{r})$ are finite. We assume that $\mathrm{g} = (\mathrm{g}_0, \mathrm{g}_1)$ is in the isotropy group of $(\mathrm{a}, \mathrm{b}, \mathrm{d}, \mathrm{e}, \mathrm{f})$. We obtain decompositions

$$\mathrm{L}_0 = \mathrm{L}_{00} \oplus \ldots \mathrm{L}_{0\mathrm{s}} \oplus \ldots \mathrm{L}_{0\mathrm{q}}$$
$$\mathrm{L}_1 = \mathrm{L}_{10} \oplus \ldots \mathrm{L}_{1\mathrm{s}} \oplus \ldots \mathrm{L}_{1\mathrm{p}}$$

with the following properties:

- $\mathrm{g}_\mathrm{i}(\mathrm{t})|_{\mathrm{L}_{\mathrm{ij}}} = \varepsilon_{\mathrm{ij}} \mathrm{id}_{\mathrm{L}_{\mathrm{ij}}}$, where the $\varepsilon_{\mathrm{ij}}$ are certain roots of unity with $\varepsilon_{\mathrm{ij}} \neq \varepsilon_{\mathrm{il}}$ for $\mathrm{j} \neq \mathrm{l}$;
- $\varepsilon_{00} = \varepsilon_{10} = 1$;
- for $\mathrm{i} = 1, \ldots, \mathrm{s}$, we have $\varepsilon_{0\mathrm{i}} = \varepsilon_{1\mathrm{i}} = \varepsilon_\mathrm{i}$;
- for all $\mathrm{i}, \mathrm{j} > \mathrm{s}$, we have $\varepsilon_{0\mathrm{i}} \neq \varepsilon_{1\mathrm{j}}$.

According to this decomposition, we may write a in blocks $\mathrm{a}_{\mathrm{ij}} : \mathrm{L}_{0\mathrm{j}} \to \mathrm{L}_{1\mathrm{i}}$. g acts on a_{ij} by multiplication with $\varepsilon_{1\mathrm{i}} \cdot \varepsilon_{0\mathrm{j}}^{-1}$. Since $\mathrm{g}_1 \mathrm{a} \mathrm{g}_0^{-1} = \mathrm{a}$, $\mathrm{a}_{\mathrm{ij}} = 0$ except in those cases, where $\mathrm{i} = \mathrm{j} \leq \mathrm{s}$. Therefore, a is of the form

$$\begin{pmatrix} \mathrm{a}_0 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \mathrm{a}_\mathrm{s} & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

with $\mathrm{a}_\mathrm{i} = \mathrm{a}_{\mathrm{ii}}$ the only not necessarily vanishing blocks, $\mathrm{i} = 1, \ldots, \mathrm{s}$. We may treat b and d in a similar fashion to obtain

$$\mathrm{b} = \begin{pmatrix} \mathrm{b}_0 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \mathrm{b}_\mathrm{s} & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \mathrm{d} = \begin{pmatrix} \mathrm{d}_0 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \mathrm{d}_\mathrm{s} & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Moreover, we can write

$$\mathrm{f} = \begin{pmatrix} \mathrm{f}_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \mathrm{e} = \begin{pmatrix} \mathrm{e}_0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is evident, that the assumptions $\mathrm{p} > \mathrm{s}$, $\mathrm{q} > \mathrm{s}$ or $\mathrm{s} > 0$ lead to the same contradictions as we have constructed in Step 5. Thus, the action is free.

Step 7. $\mathcal{Z}(\mathrm{m}, \mathrm{r}) \to \mathcal{E}(\mathrm{m}, \mathrm{r})$ *is the geometric quotient under the action of the group* $\mathrm{Gl}(\mathrm{L}_0) \times \mathrm{Gl}(\mathrm{L}_1)$.

Step 5 shows the existence of a geometric quotient. Since we can repeat the monadic construction in Theorem 2.9 locally for families, the quotient is equipped with a universal property: it is covered by local moduli of points in $\mathcal{E}(\mathrm{m}, \mathrm{r})$. Since $\mathcal{E}(\mathrm{m}, \mathrm{r})$ is represented by a fine moduli space, it is canonically isomorphic to the quotient. □

2.4 The spaces of framed exceptional jumps of order 1

Here, we use the above given monadic description to study the spaces $\mathcal{E}(1, \mathrm{r})$ in the case of k being the field of complex numbers. By Theorem 2.9 and 2.10, we are to examine all tuples $(\mathrm{a}, \mathrm{b}, \mathrm{d}, \mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}, \mathrm{f}_1, \ldots, \mathrm{f}_\mathrm{r})$ in $\mathbb{C}^3 \times \mathbb{C}^\mathrm{r} \times \mathbb{C}^\mathrm{r}$, which fulfill the equations $\mathrm{f}_1\mathrm{e}_1 + \ldots + \mathrm{f}_\mathrm{r}\mathrm{e}_\mathrm{r} = 0$, $\mathrm{bd} = 0$ and the non-degeneracy condition *(iv)*.

First, we restrict ourselves to the case of those points in $\mathcal{E}(1, \mathrm{r})$ which are non-trivial along E_1, call it the subspace $\mathcal{E}_0(1, \mathrm{r}) \subset \mathcal{E}(1, \mathrm{r})$. By Theorem 2.9 and 2.10, this space $\mathcal{E}_0(1, \mathrm{r})$ is the quotient of the space $\mathcal{Z}$ of all tuples $(\mathrm{a}, \mathrm{b}, \mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}, \mathrm{f}_1, \ldots, \mathrm{f}_\mathrm{r})$ in $\mathbb{C}^2 \times \mathbb{C}^\mathrm{r} \times \mathbb{C}^\mathrm{r}$ which fulfill the conditions

$$\begin{aligned} \mathrm{f}_1\mathrm{e}_1 + \ldots + \mathrm{f}_\mathrm{r}\mathrm{e}_\mathrm{r} &= 0, \\ (\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}) &\neq 0, \\ (\mathrm{f}_1, \ldots, \mathrm{f}_\mathrm{r}) &\neq 0, \end{aligned}$$

where the last two inequalities come from *(iv)*, by the action of $\mathbb{C}^* \times \mathbb{C}^*$

$$(\lambda, \mu)(\mathrm{a}, \mathrm{b}, \mathrm{e}, \mathrm{f}) = (\mu\lambda^{-1}\mathrm{a}, \mu\lambda^{-1}\mathrm{b}, \lambda^{-1}\mathrm{e}, \mu\mathrm{f}).$$

The supspace $\mathcal{E}_{01}(1,\mathrm{r}) \subset \mathcal{E}_0(1,\mathrm{r})$ of all framed exceptional jumps of order 1 that are non-trivial along E_0, as well as along E_1, corresponds to the additional equation $\mathrm{b} = 0$.

Proposition 2.11 *For all ranks* $\mathrm{r} \geq 2$, $\mathcal{E}_{01}(1,\mathrm{r})$ *is a strong deformation retract of* $\mathcal{E}_0(1,\mathrm{r})$.

Proof. We consider $\Phi : [0,1] \times \mathcal{Z} \to \mathcal{Z}$ with $\Phi(\mathrm{t},\mathrm{a},\mathrm{b},\mathrm{e},\mathrm{f}) = (\mathrm{a},\mathrm{tb},\mathrm{e},\mathrm{f})$. This continuous map commutes with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathcal{Z}$ because

$$(\lambda,\mu)(\Phi(\mathrm{t},\mathrm{a},\mathrm{b},\mathrm{e},\mathrm{f})) = (\mu\lambda^{-1}\mathrm{ta}, \mu\lambda^{-1}\mathrm{b}, \lambda^{-1}\mathrm{e}, \mu\mathrm{f}) = \Phi(\mathrm{t},(\lambda,\mu)(\mathrm{a},\mathrm{b},\mathrm{e},\mathrm{f})).$$

Therefore, Φ descends to a continuous map $\Phi : [0,1] \times \mathcal{E}_0(1,\mathrm{r}) \to \mathcal{E}_0(1,\mathrm{r})$. Obviously,

- Φ restricted to $\{1\} \times \mathcal{E}_0(1,\mathrm{r})$ is the identity on $\mathcal{E}_0(1,\mathrm{r})$;
- Φ restricted to $\{0\} \times \mathcal{E}_0(1,\mathrm{r})$ maps to $\mathcal{E}_{01}(1,\mathrm{r})$;
- for all $\mathrm{t} \in [0,1]$, Φ restricted to $\{\mathrm{t}\} \times \mathcal{E}_{01}(1,\mathrm{r})$ is the identity on $\mathcal{E}_{01}(1,\mathrm{r})$.

But this is just the definition of a strong deformation retract.$\square$

Corollary 2.12 *For all ranks* $\mathrm{r} \geq 2$, $\mathcal{E}_{01}(1,\mathrm{r})$ *is a strong deformation retract of* $\mathcal{E}(1,\mathrm{r})$.

Proof. By symmetry, the space $\mathcal{E}_1(1,\mathrm{r})$ of points in $\mathcal{E}(1,\mathrm{r})$ which are non-trivial along E_0 is isomorphic to $\mathcal{E}_0(1,\mathrm{r})$. The union of both is $\mathcal{E}(1,\mathrm{r})$, and their intersection is $\mathcal{E}_{01}(1,\mathrm{r})$. $\square$

Theorem 2.13 *For all ranks* $\mathrm{r} \geq 2$, $\mathcal{E}(1,\mathrm{r})$ *is homotopically equivalent to the flag variety* $\mathbb{F}_{\mathbb{C}}(1,\mathrm{r}-1) = \{(\mathrm{e},\mathrm{f}) \in \mathbb{P}_{\mathbb{C}}^{\mathrm{r}-1} \times \mathbb{P}_{\mathbb{C}}^{\mathrm{r}-1} | \mathrm{f}(\mathrm{e}) = 0\}$. *In particular, we have* $\mathcal{E}_{01}(1,2)$ *homotopically equivalent to* $\mathbb{P}_{\mathbb{C}}^1 \cong \mathrm{S}^2$.

Proof. This is just one step further than in the previous proposition. $\mathcal{E}(1,\mathrm{r})$ is by Corollary 2.12 homotopically equivalent to $\mathcal{E}_{01}(1,\mathrm{r})$, and the second is the quotient of the space $\mathcal{Z}'$ of all tuples $(\mathrm{a}, \mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}, \mathrm{f}_1, \ldots, \mathrm{f}_\mathrm{r})$ in $\mathbb{C} \times \mathbb{C}^\mathrm{r} \times \mathbb{C}^\mathrm{r}$ which fulfill the conditions

$$\mathrm{f}_1\mathrm{e}_1 + \ldots + \mathrm{f}_\mathrm{r}\mathrm{e}_\mathrm{r} = 0,$$
$$(\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{r}) \neq 0,$$
$$(\mathrm{f}_1, \ldots, \mathrm{f}_\mathrm{r}) \neq 0,$$

by the action of $\mathbb{C}^* \times \mathbb{C}^*$

$$(\lambda, \mu)(\mathrm{a}, \mathrm{e}, \mathrm{f}) = (\mu\lambda^{-1}\mathrm{a}, \lambda^{-1}\mathrm{e}, \mu\mathrm{f}).$$

We consider $\Phi : [0, 1] \times \mathcal{Z}' \to \mathcal{Z}'$ with $\Phi(\mathrm{t}, \mathrm{a}, \mathrm{e}, \mathrm{f}) = (\mathrm{ta}, \mathrm{e}, \mathrm{f})$. Obviously,

- Φ restricted to $\{1\} \times \mathcal{Z}'$ is the identity on $\mathcal{E}_0(1, \mathrm{r})$;
- Φ restricted to $\{0\} \times \mathcal{Z}'$ maps to the subspace $\mathcal{Z}''$ of tuples $(0, \mathrm{e}, \mathrm{f})$;
- for all $\mathrm{t} \in [0, 1]$, Φ restricted to $\{\mathrm{t}\} \times \mathcal{Z}''$ is the identity on $\mathcal{Z}''$.

Therefore, $\mathcal{Z}''$ is a deformation retract of $\mathcal{Z}'$.

The continuous map Φ commutes with the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathcal{Z}'$; hence, Φ descends to a continuous map $\Phi : [0, 1] \times \mathcal{E}_{01}(1, \mathrm{r}) \to \mathcal{E}_{01}(1, \mathrm{r})$, which shows that the quotient $\mathcal{Z}''/(\mathbb{C}^* \times \mathbb{C}^*)$ is a strong deformation retract of $\mathcal{E}_{01}(1, \mathrm{r})$. But this quotient is just the flag variety from the assertion. □

2.5 Finite determination

Here, *finite determination* means that an object on the exceptional local line X is already determined up to isomorphism by its restriction on the m-th infinitesimal neighbourhood $\mathrm{X}_\mathrm{n} = \mathrm{X} \times_{\mathrm{Spec A}} \operatorname{Spec} \mathrm{A}_\mathrm{n}$ of the central fibre. The ring A can be again any discrete valuation ring over k. We show that a framed exceptional jump (V, α) of order m is already determined by $(\mathrm{V}_\mathrm{m} = \mathrm{V} \otimes \mathrm{A}_\mathrm{n}, \alpha_\mathrm{m} = \alpha|_{\mathrm{X}_\mathrm{m}})$. Of course, this implies the analogous statement for framed ordinary jumps.

Lemma 2.14 *Let* (V, α) *and* (W, β) *be two exceptional framed jumps on X. If* $\mathrm{m} \geq 0$ *is an integer such that the homomorphism*

$$\mathrm{Ext}^1(\mathrm{V}, \mathrm{W}(-\mathrm{N})) \to \mathrm{Ext}^1(\mathrm{V}_\mathrm{m}, \mathrm{W}_\mathrm{m}(-\mathrm{N}))$$

induced by restriction is an isomorphism, then $(\mathrm{V_m}, \alpha_\mathrm{m}) \cong (\mathrm{W_m}, \beta_\mathrm{m})$ *is equivalent to* $(\mathrm{V}, \alpha) \cong (\mathrm{W}, \beta)$.

Proof. We consider the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{H}om(\mathrm{V}, \mathrm{m_A^{m+1}W(-N)}) & \to & \mathcal{H}om(\mathrm{V}, \mathrm{m_A^{m+1}W}) & \to & \mathcal{H}om(\mathrm{V|_N}, \mathrm{m_A^{m+1}W|_N}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{H}om(\mathrm{V}, \mathrm{W(-N)}) & \to & \mathcal{H}om(\mathrm{V}, \mathrm{W}) & \to & \mathcal{H}om(\mathrm{V|_N}, \mathrm{W|_N}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{H}om(\mathrm{V_m}, \mathrm{W_m(-N)}) & \to & \mathcal{H}om(\mathrm{V_m}, \mathrm{W_m}) & \to & \mathcal{H}om(\mathrm{V_m|_N}, \mathrm{W_m|_N}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and the resulting commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathrm{V}, \mathrm{W}) & \to & \mathrm{Hom}(\mathrm{V|_N}, \mathrm{W|_N}) & \to & \mathrm{Ext^1}(\mathrm{V}, \mathrm{W(-N)}) \\
\downarrow & & \downarrow & & \downarrow \cong \\
\mathrm{Hom}(\mathrm{V_m}, \mathrm{W_m}) & \to & \mathrm{Hom}(\mathrm{V_m|_N}, \mathrm{W_m|_N}) & \to & \mathrm{Ext^1}(\mathrm{V_m}, \mathrm{W_m(-N)})
\end{array}$$

A diagram chase remains: Let γ_m be the image of the isomorphism $\gamma = \beta^{-1} \circ \alpha$ in $\mathrm{Hom}(\mathrm{V_m|_N}, \mathrm{W_m|_N})$. With the assumption $(\mathrm{V_m}, \alpha_\mathrm{m}) \cong (\mathrm{W_m}, \beta_\mathrm{m})$, there is a preimage of γ_m in $\mathrm{Hom}(\mathrm{V_m}, \mathrm{W_m})$. In particular, the image of γ_m in $\mathrm{Ext^1}(\mathrm{V_m}, \mathrm{W_m(-N)})$ vanishes. Because of the isomorphism between the modules $\mathrm{Ext^1}(\mathrm{V}, \mathrm{W(-N)})$ and $\mathrm{Ext^1}(\mathrm{V_m}, \mathrm{W_m(-N)})$, the image of γ in $\mathrm{Ext^1}(\mathrm{V}, \mathrm{W(-N)})$ vanishes, too. Hence, we find a preimage γ' of γ in $\mathrm{Hom}(\mathrm{V}, \mathrm{W})$. As it is a morphism between framed jumps different from zero, γ' is an isomorphism. □

Corollary 2.15 *Up to isomorphy, the spaces* $\mathcal{J}_\mathrm{A}(\mathrm{m}, \mathrm{r})$ *and* $\mathcal{E}_\mathrm{A}(\mathrm{m}, \mathrm{r})$ *do not depend on the discrete valuation ring* A *over* k.

Proof. If (V, α) and (W, β) are two framed exceptional jumps, then $\mathcal{H}om(\mathrm{V}, \mathrm{W})$ is a jump, too. The A-module $\mathrm{Ext^1}(\mathrm{V}, \mathrm{W(-N)})$ is hence of finite length equal to

$\mathrm{m} = \mathrm{ord}(\mathcal{H}om(\mathrm{V}, \mathrm{W}))$. Because of base change in the highest relative dimension, we have

$$\begin{aligned}\mathrm{Ext}^1(\mathrm{V}, \mathrm{W}(-\mathrm{N})) = \mathrm{R}^1\pi_*\mathcal{H}om(\mathrm{V}, \mathrm{W})(-\mathrm{N}) &= (\mathrm{R}^1\pi_*\mathcal{H}om(\mathrm{V}, \mathrm{W})(-\mathrm{N})) \otimes \mathrm{A}_\mathrm{m} \\ &= \mathrm{Ext}^1(\mathrm{V}_\mathrm{m}, \mathrm{W}_\mathrm{m}(-\mathrm{N}))\end{aligned}$$

and infer from Lemma 2.14 that framed jumps are already determined by their restriction to an infinitesimal neighbourhood of the central fibre. Since the formal neighbourhood of the central fibre does not depend, up to isomorphism, on the fixed discrete valuation ring A, we have obtained our corollary. □

In particular, the statements of the Sections 2.2 - 2.4 can be applied for other cases than $\mathrm{A} = \mathrm{k}[\mathrm{t}]_{(\mathrm{t})}$. This is true, for instance, for the given monadic descriptions as well as for the fact that the condition $\mathrm{ord}_0 = 1$ is dense in $\mathcal{J}_\mathrm{A}(\mathrm{m}, \mathrm{r})$ and $\mathcal{E}_\mathrm{A}(\mathrm{m}, \mathrm{r})$.

Although we will prove the finite determination below with an explicit description of the necessary infinitesimal neighbourhood only for framed exceptional jumps, it is quite obvious by the proofs of Lemma 2.14 and Corollary 2.15 that finite determination on some infinitesimal neighbourhood also holds for framed jumps on more exotic local lines obtained by blowing up Y in an arbitrary centre situated on the central line.

Lemma 2.16 *Let* V *and* W *be exceptional jumps on* X *which are both non-trivial along* E_i *with* $\mathrm{i} \in \{0, 1\}$ *fixed. If the annulator of the A-module* $\mathrm{R}^1\pi_*\mathrm{V}(-\mathrm{N})$ *is* $\mathrm{m}_\mathrm{A}^{\mathrm{m}+1}$ *and the annulator of the A-module* $\mathrm{R}^1\pi_*\mathrm{W}(-\mathrm{N})$ *is* $\mathrm{m}_\mathrm{A}^{\mathrm{n}+1}$, *then the annulator of the* A-*module* $\mathrm{Ext}^1(\mathrm{V}, \mathrm{W}(-\mathrm{N}))$ *is contained in the ideal* $\mathrm{m}_\mathrm{A}^{\max\{\mathrm{m},\mathrm{n}\}+1}$.

Proof. Without loss of generality, we assume $\mathrm{i} = 0$. The claim is obviously true in the case that V and W are pullbacks under σ_0 of ordinary jumps with the property that $\mathrm{V}_\mathrm{m} = \mathrm{V}_0 \otimes_\mathrm{k} \mathrm{A}_\mathrm{m}$ and $\mathrm{W}_\mathrm{n} = \mathrm{W}_0 \otimes_\mathrm{k} \mathrm{A}_\mathrm{n}$. This property is open and dense in families of pairs of pullbacks of ordinary jumps by Corollary 2.6. In particular, it is fulfilled whenever $\mathrm{ord}_0\mathrm{V} = \mathrm{ord}_0\mathrm{W} = 1$, $\mathrm{ord}\,\mathrm{V} = \mathrm{m} + 1$ and $\mathrm{ord}\,\mathrm{W} = \mathrm{n} + 1$. On the other side, the property $\mathrm{ann}_\mathrm{A}\mathrm{Ext}^1(\mathrm{V}, \mathrm{W}(-\mathrm{N})) \subset \mathrm{m}_\mathrm{A}^{\max\{\mathrm{m},\mathrm{n}\}+1}$ is closed. Hence, the claim is true for all pullbacks of pairs of ordinary jumps under σ_0.

To be such a pullback is again an open and dense property in families of exceptional jumps which are non-trivial along E_0. Indeed, Step 4 of the proof of Theorem 2.10 shows that $\sigma_0^*\mathcal{J}(\mathrm{m},\mathrm{r})$ is dense in $\mathcal{E}(\mathrm{m},\mathrm{r}) - \sigma_1^*\mathcal{J}(\mathrm{m},\mathrm{r})$. Therefore the lemma is true in general. □

Theorem 2.17 *(Finite Determination.) Let* (V,α) *and* (W,β) *be two exceptional framed jumps of order* $\mathrm{m}+1$. *We have*

(i) $(\mathrm{V},\alpha) \cong (\mathrm{W},\beta)$ *if and only if* $(\mathrm{V}_\mathrm{m},\alpha_\mathrm{m}) \cong (\mathrm{W}_\mathrm{m},\beta_\mathrm{m})$,

(ii) $\mathrm{V} \cong \mathrm{W}$ *if and only if* $\mathrm{V}_\mathrm{m} \cong \mathrm{W}_\mathrm{m}$.

Proof. Lemma 2.16 implies $\mathrm{ann}_\mathrm{A}\mathrm{Ext}^1(\mathrm{V},\mathrm{W}(-\mathrm{N})) \subset \mathrm{m}_\mathrm{A}^{\mathrm{m}+1}$. Because of base change in the highest dimension, we have the module $\mathrm{Ext}^1(\mathrm{V},\mathrm{W}(-\mathrm{N})) \otimes \mathrm{A}_\mathrm{m}$ equal to $\mathrm{Ext}^1(\mathrm{V}_\mathrm{m},\mathrm{W}_\mathrm{m}(-\mathrm{N}))$, and infer $\mathrm{Ext}^1(\mathrm{V},\mathrm{W}(-\mathrm{N})) = \mathrm{Ext}^1(\mathrm{V}_\mathrm{m},\mathrm{W}_\mathrm{m}(-\mathrm{N}))$. Thus, the assumption of Lemma 2.14 is fulfilled, and we obtain (i).

The second claim (ii) is a consequence from (i) as follows: For a given isomorphism $\gamma_\mathrm{m} : \mathrm{V}_\mathrm{m} \cong \mathrm{W}_\mathrm{m}$ and an arbitrary framing $\alpha_\mathrm{m} : \mathrm{V}_\mathrm{m}|_\mathrm{N} \cong \mathcal{O}_\mathrm{N}^\mathrm{r} \otimes \mathrm{A}_\mathrm{m}$, we just choose the framing $\beta_\mathrm{m} = \gamma_\mathrm{m}^{-1} \circ \alpha_\mathrm{m}$ and arbitrary lifts α and β. □

2.6 The tangent map of $\mathcal{V}(\mathrm{m},\mathrm{r}) \to \mathrm{Sym}^{\mathrm{m}}\mathrm{N}$

Consider again $\mathrm{A} = \mathrm{k}[\mathrm{t}]_{(\mathrm{t})}$, and let M be an A-module of finite length m. If we define $\mathrm{T}_\mathrm{i} = \mathrm{k}[\mathrm{t}]/(\mathrm{t})^{\mathrm{i}+1}$, we may assume that $\mathrm{M} = \mathrm{T}_{\mathrm{m}_1} \oplus \ldots \oplus \mathrm{T}_{\mathrm{m}_\mathrm{p}}$ with $\mathrm{m}_1 + \ldots + \mathrm{m}_\mathrm{p} = \mathrm{m}$. We choose a free resolution of M

$$0 \to \mathrm{A}^\mathrm{p} \xrightarrow{\mathrm{s}} \mathrm{A}^\mathrm{p} \xrightarrow{\mathrm{b}} \mathrm{M} \to 0$$

with

$$\mathrm{s} = \begin{pmatrix} \mathrm{t}^{\mathrm{m}_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{t}^{\mathrm{m}_\mathrm{p}} \end{pmatrix}$$

and obtain the determinantal divisor of M as $\mathrm{div}(\det \mathrm{s})$.

Now consider $(V, \alpha) \in \mathcal{V}(m, r)(\operatorname{Spec} k)$. We look for the tangent map

$$T_{(V,\alpha)}\mathcal{V}(m, r) \to T_D \mathrm{Sym}^m N,$$

where D is the determinantal divisor belonging to $R^1\pi_*V(-N)$. We assume that $D = \operatorname{div}(t^m)$ and $R^1\pi_*V(-N) = M$. Then D is also equal to the determinantal divisor of M.

A tangent vector θ at (V, α) corresponds to a deformation of (V, α) over $\operatorname{Spec} k[\varepsilon]$. This deformation yields an element $e \in \operatorname{Ext}^1_A(M, M)$ represented by a short exact sequence

$$0 \to M \to \tilde{M} \to M \to 0.$$

We choose the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A^p & \xrightarrow{s} & A^p & \xrightarrow{b} & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A^{2p} & \xrightarrow{\tilde{s}} & A^{2p} & \xrightarrow{\tilde{b}} & \tilde{M} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A^p & \xrightarrow{s} & A^p & \xrightarrow{b} & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We may consider $\tilde{M}$ and A^{2p} as $k[\varepsilon]$-modules. Under this point of view we obtain $\tilde{s} = s + \varepsilon s'$ and

$$\det \tilde{s} = t^m + \varepsilon \sum s'_{ii} t^{m_1 + \ldots + \check{m}_i + \ldots + m_p} = t^m + \varepsilon\sigma.$$

Hereby, $\check{m}_i$ means the ommiting of the i-th summand. Evidently, the element $\sigma \in A$ can be a unit only if $p = 1$. The line bundle over $\operatorname{Spec} A[\varepsilon]$ associated to $\operatorname{div}(\det \tilde{s})$ corresponds to the module $(1/(t^m + \varepsilon\sigma))A[\varepsilon]$, and the associated vector in $(1/t^{-m})A/A$, the tangent space at D, is precisely $t^{-m}\sigma$ modulo 1.

We have thus obtained:

Theorem 2.18 $\Delta : \mathcal{V}(\mathrm{m},\mathrm{r}) \to \mathrm{Sym}^{\mathrm{m}}\mathrm{N}$ *is smooth only in those points* (V,α)*, where* $\dim_{\mathrm{k}}\mathrm{H}^1(\mathrm{F}_{\mathrm{P}}, \mathrm{V}|_{\mathrm{F}_{\mathrm{P}}}(-\mathrm{N})) \leq 1$ *for all points* $\mathrm{P} \in \mathrm{N}$*. The smooth part of* $\mathcal{J}(\mathrm{m},\mathrm{r})$ *is given by the condition* $\mathrm{ord}_0 = 1$*.*

Of course, the occurence of singularities is caused by the fact that Δ completely forgets about the framing. It is tempting to amend this situation by introducing the notion of a *framed determinantal divisor*, which could be done in several ways. Alas, the spaces of such framed determinantal divisors are necessarily much more involved and less familiar than $\mathrm{Sym}^{\mathrm{m}}\mathrm{N}$, which destroys the benefit of the whole action and proves another time the Theorem of Conservation of Difficulty.

3 On the topology of moduli of instantons

3.1 Homological stability

As promised in the introduction, we prove in this section the following theorem:

Theorem 3.1 *Let* M *be a self-dual oriented Riemannian 4-dimensional manifold with twistor fibration* $\mathrm{P} \to \mathrm{M}$*, such that P contains a surface of degree 1, which in turn contains the twistor fibre over a point* $\mathrm{x} \in \mathrm{M}$*. Let* $\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r})$ *denote the moduli space of based* $\mathrm{SU(r)}$*-instantons of charge* m *on* (M,x) *with* $\mathrm{r} \geq 2$ *an arbitrary rank. There is a stabilisation map* $\iota : \mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r}) \to \mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}+1,\mathrm{r})$ *such that the induced homomorphisms* $\mathrm{H}_{\mathrm{t}}(\iota) : \mathrm{H}_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r})) \to \mathrm{H}_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}+1,\mathrm{r}))$ *on the homology groups are isomorphisms for all* $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$*. In the case of* $\mathrm{M} = \mathrm{S}^4$*,* ι *is homotopy equivalent to the original stabilisation map of Taubes ([39]) and the ones used in [4] and [40].*

Throughout this section, we consider homology groups with coefficients in a fixed group, which may be assumed to be $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$.

By Theorem 1.5 and 1.10, the moduli space $\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r})$ is real-analytically isomorphic to a moduli space $\mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r})$. Recall from Section 1.3 that thereby $\pi : \mathrm{S} \to \mathbb{P}^1_{\mathbb{C}}$ is the blowing up of the first Hirzebruch surface $\tilde{\mathbb{P}}^2_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ in n distinct points, of which no two are situated on the same fibre over $\mathbb{P}^1$. N is a section of π. For $\mathrm{z} \in \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathrm{F_z}$ is the fibre of π over z, and we assume that F_∞ is smooth. We denote with $\mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r})$ the moduli space of framed vector bundles (V,α), where V is a rank-r vector bundle on S with trivial determinant and second Chern class equal to m, which is trivial along $\mathrm{N} \cup \mathrm{F}_\infty$, and where $\alpha : \mathrm{V}|_{\mathrm{N}} \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{N}}$.

Lemma 3.2 *For* n *fixed, all the manifolds* $\mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r})$ *are homotopy equivalent and even homeomorphic.*

Proof. To see this, we consider two such surfaces S and S′ with exceptional fibres over $\mathrm{P}_1,\ldots,\mathrm{P}_\mathrm{n}$ and $\mathrm{P}'_1,\ldots,\mathrm{P}'_\mathrm{n}$, respectively. We choose any homeomorphic homotopy equivalence $(\mathbb{C},\mathrm{P}_1,\ldots,\mathrm{P}_\mathrm{n}) \simeq (\mathbb{C},\mathrm{P}'_1,\ldots,\mathrm{P}'_\mathrm{n})$ of n-pointed spaces, that is a continuous map $\mathrm{h} : \mathbb{C} \times [0,1] \to \mathbb{C}$ such that all restrictions $\mathrm{h_t} = \mathrm{h}|_{\mathbb{C}\times\{\mathrm{t}\}}$ are homeomorphisms, h_0 is the identity and h_1 maps $\mathrm{P_i}$ to P'_i.

Because of Corollary 1.9, this homotopy equivalence lifts along the morphisms provided by the determinantal divisor to a commutative diagram:

$$\begin{array}{ccc} \mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r}) & \simeq & \mathcal{V}(\mathrm{S}',\mathrm{m},\mathrm{r}) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^{\mathrm{m}}(\mathbb{C},\mathrm{P}_1,\ldots,\mathrm{P}_\mathrm{n}) & \simeq & \mathrm{Sym}^{\mathrm{m}}(\mathbb{C},\mathrm{P}'_1,\ldots,\mathrm{P}'_\mathrm{n}) \end{array} \quad \square$$

We denote with $\mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r})$ an arbitrarily fixed representative $\mathcal{V}(\mathrm{S},\mathrm{m},\mathrm{r})$ with the property that the exceptional fibres of S over $\mathbb{P}^1_{\mathbb{C}}$ are $\mathrm{F}_1,\ldots,\mathrm{F}_\mathrm{n}$, where we consider the integers as elements in $\mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}}$.

For positive integers s, we let $\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m},\mathrm{r})$ be the open submanifold of $\mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r})$ corresponding to all points (V,α) with $\mathrm{V}|_{\mathrm{F_i}}$ trivial for $\mathrm{i} \in \{1,\ldots,\mathrm{s}\} \subset \mathbb{Z} \subset \mathbb{P}^1_{\mathbb{C}}$, and put $\mathcal{V}_0(\mathrm{n},\mathrm{m},\mathrm{r}) = \mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r})$. Moreover, for X any subspace of $\mathbb{C}$, we define $\mathcal{V}^{\mathrm{X}}_\mathrm{s}(\mathrm{n},\mathrm{m},\mathrm{r})$ to be the subspace of $\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m},\mathrm{r})$ consisting of all points (V,α), where V is trivial along all fibres of π over $\mathbb{P}^1_{\mathbb{C}} - \mathrm{X}$.

Lemma 3.3 *If B is a simply connected open domain in* $\mathbb{C}$ *containing the integers* $\{1,\ldots,\max(\mathrm{s},\mathrm{n})\}$, *then* $\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m},\mathrm{r})$ *is homotopy equivalent, and in fact homeomorphic, to its open subspace* $\mathcal{V}^{\mathrm{B}}_\mathrm{s}(\mathrm{n},\mathrm{m},\mathrm{r})$.

Proof. This is done analogously to the previous lemma by considering a homeomorphic homotopy equivalence between $\mathbb{C}$ and B which leaves $1, \ldots, \max(\mathrm{s}, \mathrm{n})$ fixed. $\square$

Now, we fix B as the open disk around 0 of radius $\mathrm{r}_0 > \mathrm{n}$ and choose a point $\mathrm{z}_0 \in \mathbb{P}^1_{\mathbb{C}} - (\mathrm{B} \cup \{\infty\})$ and a framed ordinary jump (V_0, α_0) of order 1 and rank r. We obtain a continuous map

$$\iota : \mathcal{V}(\mathrm{n}, \mathrm{m}, \mathrm{r}) \cong \mathcal{V}_{\mathrm{B}}(\mathrm{n}, \mathrm{m}, \mathrm{r}) \to \mathcal{V}(\mathrm{n}, \mathrm{m} + 1, \mathrm{r})$$

by adding to each $(\mathrm{V}, \alpha) \in \mathcal{V}_{\mathrm{B}}(\mathrm{n}, \mathrm{m}, \mathrm{r})$ the framed jump (V_0, α_0) at the point z_0 via the cutting and gluing procedure described in Section 1.3. This ι is our candidate for the *stabilisation map*. Up to homotopy equivalence, it does not depend on the chosen r_0. In the case $\mathrm{n} = 0$, it coincides with the stabilisation map used in [4] and [40], and by [4, Lemma 1.22], it is homotopic to the stabilisation map of Taubes.

The above Theorem 3.1 is now a corollary of the following.

Theorem 3.4

$$\mathrm{H}_{\mathrm{t}}(\iota) : \mathrm{H}_{\mathrm{t}}(\mathcal{V}(\mathrm{n}, \mathrm{m}, \mathrm{r})) \to \mathrm{H}_{\mathrm{t}}(\mathcal{V}(\mathrm{n}, \mathrm{m} + 1, \mathrm{r}))$$

is an isomorphism for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$.

The proof of this theorem is the object of Section 3.3.

3.2 L-stratified maps

Let

$$\mathrm{M}_0 \hookrightarrow \ldots \hookrightarrow \mathrm{M}_\ell = \mathrm{M}$$

be an increasing filtration of the smooth manifold M by open-dense submanifolds such that $\{\mathrm{S}_{\mathrm{i}} = \mathrm{M}_{\mathrm{i}} - \mathrm{M}_{\mathrm{i}-1}\}$ is a stratification of M by smooth submanifolds, where the normal bundles $\mathcal{N}_{\mathrm{S}_{\mathrm{i}}|\mathrm{M}}$ are all orientable. Such a stratification $\mathrm{M} = \coprod \mathrm{S}_{\mathrm{i}}$ will be called an *L-stratification*. We denote with d_{i} the codimension of S_{i} in M. This notion of an L-stratification was taken over from [4], and will be crucial for the proof of Theorem 3.4. Examples are provided by the following lemma.

Lemma 3.5 *Let* $\mathrm{M} = \coprod_{\mathrm{i}=0,\ldots,\ell} \mathrm{S_i}$ *be a stratification of a complex manifold by locally closed submanifolds such that all* $\mathrm{M_i} = \coprod_{\mathrm{p}=0,\ldots,\mathrm{i}} \mathrm{S_p}$ *are open and dense submanifolds of M. Then it is an L-stratification.*

Proof. The normal bundles of complex submanifolds are holomorphic bundles and therefore orientable. □

The benefit of such an L-stratification is explained in [4, Theorem 5.10]. There, it is proved that the homology spectral sequence associated to the filtration $\mathrm{M}_\bullet$ starts at the E^1-level with

$$\mathrm{E}^1_{\mathrm{p,q}} = \mathrm{H}_{\mathrm{p+q}}(\Sigma^{\mathrm{d_p}}(\mathrm{S_p} \sqcup *)),$$

where $\mathrm{S_p} \sqcup *$ is the disjoint union of $\mathrm{S_p}$ with the one-point space $*$, producing a based space, and $\Sigma^{\mathrm{d_p}}$ is the $\mathrm{d_p}$-fold reduced suspension. This spectral sequence abuts upon a filtration of $\mathrm{H}_\bullet(\mathrm{M})$. We will use the equivalent statement made in the next theorem.

Theorem 3.6 *For* $\mathrm{M} = \coprod \mathrm{S_i}$ *an L-stratification as above with corresponding filtration* $\mathrm{M}_\bullet$*, the homology spectral sequence associated to* $\mathrm{M}_\bullet$ *starts with* E^1*-terms*

$$\mathrm{E}^1_{\mathrm{p,q}} = \mathrm{H}_{\mathrm{p+q-d_p}}(\mathrm{S_p}).$$

As the filtration is finite, this spectral sequence converges to a filtration of $\mathrm{H}_\bullet(\mathrm{M})$*.*

Proof. Of course, this result can be obtained as a corollary of [4, Theorem 5.10], by realising that $\mathrm{H}_{\mathrm{p+q-d_p}}(\mathrm{S_p}) = \mathrm{H}_{\mathrm{p+q}}(\Sigma^{\mathrm{d_p}}(\mathrm{S_p} \sqcup *))$. But since it is our strongest tool to showing Theorem 3.4, we give here an extra, self-contained proof.

The filtration $\mathrm{M}_\bullet$ induces a filtration on the complex of singular chains of M. Associated to this filtration, there is a spectral sequence converging to a filtration of $\mathrm{H}_\bullet(\mathrm{M})$ with E^1-terms

$$\mathrm{E}^1_{\mathrm{p,q}} = \mathrm{H}_{\mathrm{p+q}}(\mathrm{M_p}, \mathrm{M_{p-1}}),$$

which is just the known homology spectral sequence of a filtration.

By the excision axiom, we have $H_{p+q}(M_p, M_{p-1}) = H_{p+q}(M_p - U, M_{p-1} - U)$ for any open $U \subset M$ with $\bar{U} \subset M_{p-1}$. We choose T as a closed tubular neighbourhood of $S_p = M_p - M_{p-1}$ and $U = M_p - T$, and obtain $H_{p+q}(M_p, M_{p-1}) = H_{p+q}(T, T - S_p)$.

In the notation of [38], the pair $(T, T - S_p)$ is a $(d_p - 1)$-sphere bundle over S_p, where d_p is again the codimension of the stratum S_p in M. The tubular neighbourhood T is homotopy equivalent to the normal bundle of S_p in M, which can be shown, as usual, with the exponential function. By assumption, the sphere bundle $(T, T - S_p)$ is therefore orientable, and we can apply the Thom isomorphism theorem ([38, Theorem 5.7.10]) to obtain $H_{p+q}(T, T - S_p) = H_{p+q-d_p}(S_p)$. $\square$

For two L-stratified manifolds $M_0 \hookrightarrow \ldots \hookrightarrow M_\ell = M$ and $M'_0 \hookrightarrow \ldots \hookrightarrow M'_{\ell'} = M'$, a continuous map $f : M \to M'$ is said to be *L-stratified* if it preserves the filtration. Such an L-stratified map induces of course a map of the associated homology Leray spectral sequences. We also note that necessarily $\ell \leq \ell'$. The easiest example of an L-stratified map is obtained if we consider for a given L-stratification

$$M_0 \hookrightarrow \ldots \hookrightarrow M_\ell \hookrightarrow \ldots \hookrightarrow M_{\ell+\ell'} = M$$

the inclusion $M_\ell \hookrightarrow M$. In this situation, the following lemma holds.

Lemma 3.7 *If* $d_p > t_0 + 1$ *for all* $p > \ell$ *and a fixed integer* t_0*, then the inclusion* $M_\ell \hookrightarrow M$ *induces isomorphisms in the homology* $H_t(M_\ell) \cong H_t(M)$ *for all* $t \leq t_0$.

Proof. By evident induction, it is enough to show the assertion in the case $\ell' = 1$. We define a new L-stratification on M consisting only of the two strata $S'_0 = M_\ell$ and $S'_1 = S_{\ell+1} = M - M_\ell$. Due to Theorem 3.6, the $E^1_{p,q}$ terms in the associated homology Leray spectral sequence vanish for $p \neq 1, 2$. Recall, that the differentials d^r_{pq} of a spectral sequence go from E^r_{pq} to $E^r_{p-r,q+r-1}$. Our spectral sequence therefore degenerates at E^2, i.e. $H_t(M) = \oplus_{p+q=t} E^2_{p,q}$. As the codimension of the stratum S'_1 is greater than $t_0 + 1$, we also have $E^1_{1,q} = 0$ for $q \leq t_0$. Hence, $H_t(M) = E^1_{0,t} = H_t(M_\ell)$ for all $t \leq t_0$. $\square$

Lemma 3.8 *Consider a fixed integer* t_0 *and an L-stratified map*

$$f : \left(M = \coprod_{p=0,\dots,\ell} S_p \right) \longrightarrow \left(M' = \coprod_{p=0,\dots,\ell'} S'_p \right)$$

with d_p *and* d'_p *the codimensions of* S_p *and* S'_p *in* M *and* M'*, respectively. If*

- $d_p \leq d'_p$ *for all* $p = 0, \dots, \ell$*;*
- $d'_p > t_0 + 1$ *for all* $\ell < p \leq \ell'$*;*
- $H_t(f) : H_t(S_p) \to H_t(S'_p)$ *are isomorphisms for all* $p = 0, \dots, \ell$ *and for all* $t \leq t_0 + p - d_p$*;*

then $H_t(f) : H_t(M) \to H_t(M')$ *are isomorphisms for all* $t \leq t_0$*.*

Proof. Because of Lemma 3.7 and because of the assumption that $d'_p > t_0 + 1$ for all $\ell < p \leq \ell'$, we may assume that $\ell = \ell'$.

By Theorem 3.6, the homology spectral sequence $E^\bullet(M_\bullet)$ associated to $M_\bullet$ starts with $E^1_{p,q} = H_{p+q-d_p}(S_p)$. The first table of $E^\bullet(M_\bullet)$ looks like

$$\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
E^1_{2,-2}(M_\bullet) & E^1_{2,-1}(M_\bullet) & E^1_{2,0}(M_\bullet) & E^1_{2,1}(M_\bullet) & E^1_{2,2}(M_\bullet) & E^1_{2,3}(M_\bullet) & \dots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & E^1_{1,-1}(M_\bullet) & E^1_{1,0}(M_\bullet) & E^1_{1,1}(M_\bullet) & E^1_{1,2}(M_\bullet) & E^1_{1,3}(M_\bullet) & \dots \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & & E^1_{0,0}(M_\bullet) & E^1_{0,1}(M_\bullet) & E^1_{0,2}(M_\bullet) & E^1_{0,3}(M_\bullet) & \dots
\end{array}$$

where the arrows correspond to the differentials d^1_{pq}. The first table of $E^\bullet(M'_\bullet)$ looks alike. By the assumption, f induces a map $E^\bullet f : E^\bullet(M_\bullet) \to E^\bullet(M'_\bullet)$ of spectral sequences and in particular isomorphisms $E^1_{p,q}(M_\bullet) \cong E^1_{p,q}(M'_\bullet)$ for $q \leq t_0$. Because of the naturality of the differentials, we obtain $E^2_{p,q}(M_\bullet) \cong E^2_{p,q}(M'_\bullet)$ for $q \leq t_0$. Remembering that $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$, we obtain $E^3_{p,q}(M_\bullet) \cong E^3_{p,q}(M'_\bullet)$ for $q \leq t_0 - 1$ and for those (p, q) with $q = t_0$ and $p \geq 2$. Successively, we obtain that $E^r_{p,q}(M_\bullet) \cong E^r_{p,q}(M'_\bullet)$ for all r and all $p + q \leq t_0$. Because of the convergence of both spectral sequences, we have proved the lemma. □

The previous lemma provides a means to deduce from the homological behaviour of an L-stratified map on the strata its homological behaviour on the whole spaces. Now, with the next lemma we want to do it the other way around.

Lemma 3.9 *Consider a fixed integer* t_0 *and an L-stratified map*

$$\mathrm{f}:\left(\mathrm{M}=\coprod_{\mathrm{p}=0,\dots,\ell}\mathrm{S}_\mathrm{p}\right)\longrightarrow\left(\mathrm{M}'=\coprod_{\mathrm{p}=0,\dots,\ell'}\mathrm{S}'_\mathrm{p}\right)$$

with d_p *and* d'_p *the codimensions of* S_p *and* S'_p *in* M *and* M'*, respectively. If*

- $\mathrm{d}_\mathrm{p}\le\mathrm{d}'_\mathrm{p}$ *for all* $\mathrm{p}=0,\dots,\ell$*;*
- $\mathrm{d}'_\mathrm{p}>\mathrm{t}_0+1$ *for all* $\ell<\mathrm{p}\le\ell'$*;*
- $\mathrm{H}_\mathrm{t}(\mathrm{f}):\mathrm{H}_\mathrm{t}(\mathrm{S}_\mathrm{p})\to\mathrm{H}_\mathrm{t}(\mathrm{S}'_\mathrm{p})$ *are isomorphisms for all* $\mathrm{p}=1,\dots,\ell$ *and for all* $\mathrm{t}\le\mathrm{t}_0+\mathrm{p}-\mathrm{d}_\mathrm{p}$*;*
- $\mathrm{H}_\mathrm{t}(\mathrm{f}):\mathrm{H}_\mathrm{t}(\mathrm{M})\to\mathrm{H}_\mathrm{t}(\mathrm{M}')$ *are isomorphisms for all* $\mathrm{t}\le\mathrm{t}_0$*;*

then $\mathrm{H}_\mathrm{t}(\mathrm{f}):\mathrm{H}_\mathrm{t}(\mathrm{M}_0)\to\mathrm{H}_\mathrm{t}(\mathrm{M}'_0)$ *are isomorphisms for all* $\mathrm{t}\le\mathrm{t}_0$*.*

Proof. Because of Lemma 3.7, we may assume that $\ell=\ell'$. The statement is moreover obvious for $\ell=0$. We define new L-stratifications on M and M′ by putting $\mathrm{M}=\mathrm{M}_{\ell-1}\sqcup\mathrm{S}_\ell$ and $\mathrm{M}'=\mathrm{M}'_{\ell-1}\sqcup\mathrm{S}'_\ell$. Due to our assumptions, a quick look at the E^1-table of the associated homology Leray spectral sequences gives us, analogous to the proof of Lemma 3.7, the following commutative diagram for all $\mathrm{t}\le\mathrm{t}_0$:

$$\begin{array}{ccc}
\mathrm{H}_\mathrm{t}(\mathrm{M}_{\ell-1}) & \xrightarrow{\mathrm{H}_\mathrm{t}\mathrm{f}} & \mathrm{H}_\mathrm{t}(\mathrm{M}'_{\ell-1}) \\
\downarrow = & & \downarrow = \\
\mathrm{H}_\mathrm{t}(\mathrm{M}) & \xrightarrow{\mathrm{H}_\mathrm{t}\mathrm{f}} & \mathrm{H}_\mathrm{t}(\mathrm{M}')
\end{array}$$

Thus, it is enough to show the lemma for $\mathrm{f}:\mathrm{M}_{\ell-1}\to\mathrm{M}'_{\ell-1}$, and the proof is completed via induction. □

In the next subsection, we will also use the following lemma.

Lemma 3.10 *Let* X, X' *and* Y *be topological spaces,* $\mathrm{f}:\mathrm{X}\to\mathrm{X}'$ *a continuous map and* t_0 *an integer. If* $\mathrm{H}_\mathrm{t}\mathrm{f}$ *is an isomorphism for all* $\mathrm{t}\le\mathrm{t}_0$*, then the same is true for*

$$\mathrm{H}_\mathrm{t}(\mathrm{f}\times\mathrm{id}_\mathrm{Y}):\mathrm{H}_\mathrm{t}(\mathrm{X}\times\mathrm{Y})\to\mathrm{H}_\mathrm{t}(\mathrm{X}'\times\mathrm{Y}).$$

Proof. Due to the theorem of Künneth, there is a natural short exact sequence

$$\begin{array}{ccccccccc}
0 & \to & \bigoplus\limits_{\mathrm{p+q=t}} \mathrm{H_pX}\otimes\mathrm{H_qY} & \longrightarrow & \mathrm{H_t(X\times Y)} & \longrightarrow & \bigoplus\limits_{\mathrm{p+q=t-1}} \mathrm{Tor_1(H_pX, H_qY)} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus\limits_{\mathrm{p+q=t}} \mathrm{H_pX'}\otimes\mathrm{H_qY} & \longrightarrow & \mathrm{H_t(X'\times Y)} & \longrightarrow & \bigoplus\limits_{\mathrm{p+q=t-1}} \mathrm{Tor_1(H_pX', H_qY)} & \to & 0
\end{array}$$

where the first and the third downward arrows are isomorphisms for $\mathrm{t}\le\mathrm{t}_0$, wherewith the middle downward arrow is an isomorphism, too. $\square$

3.3 Proof of Theorem 3.4

Let $\mathrm{s}\ge 0$ be a fixed integer. In the definition of the stabilisation map ι after Lemma 3.3, we may choose $\mathrm{r}_0>\mathrm{s}$. Because of Lemma 3.3, we may thus assume that ι maps $\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m},\mathrm{r})$ to $\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m}+1,\mathrm{r})$.

Proposition 3.11

$$\mathrm{H}_\mathrm{t}(\iota):\mathrm{H}_\mathrm{t}(\mathcal{V}_\mathrm{s}(0,\mathrm{m},\mathrm{r}))\to\mathrm{H}_\mathrm{t}(\mathcal{V}_\mathrm{s}(0,\mathrm{m}+1,\mathrm{r}))$$

is an isomorphism for all $\mathrm{t}\le\frac{\mathrm{m}}{2}-1$.

Proof. We will proceed by induction on s and m. For $\mathrm{s}=0$, the claim holds by the results of [4] and [40], and it moreover holds for trivial reasons in the case of $\mathrm{m}=0$. We assume now that our assertion is true for the stabilisation maps $\iota:\mathcal{V}_{\mathrm{s}'}(0,\mathrm{m}',\mathrm{r})\to\mathcal{V}_{\mathrm{s}'}(\mathrm{n},\mathrm{m}'+1,\mathrm{r})$ with $\mathrm{s}'<\mathrm{s}$ or $\mathrm{m}'<\mathrm{m}$.

We define $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(\mathrm{p})$ to be the subspace of $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})$ consisting of all points (V,α), where V has a jump of order no greater than p at F_s. Because of the Semicontinuity Theorem,

$$\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(0)\hookrightarrow\ldots\hookrightarrow\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(\mathrm{m})=\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})$$

is an increasing filtration by open and dense submanifolds. We also have that ι maps the stratum

$$\mathcal{V}_{s-1}(n, m, r)(p) - \mathcal{V}_{s-1}(n, m, r)(p-1)$$

to the stratum

$$\mathcal{V}_{s-1}(n, m+1, r)(p) - \mathcal{V}_{s-1}(n, m+1, r)(p-1).$$

Alas, the stratum $\mathcal{V}_{s-1}(n, m, r)(p) - \mathcal{V}_{s-1}(n, m, r)(p-1)$ is isomorphic to the variety $\mathcal{V}_s(n, m-p, r) \times \mathcal{J}(p, r)$, and therefore non-smooth for $p > 1$. Hence, to obtain an L-stratification we have to refine our considerations.

We take $\mathcal{J}(p, r)(1)$ as the open regular part of the variety $\mathcal{J}(p, r)$ and define inductively $\mathcal{J}(p, r)(q)$ as the open regular part of $\mathcal{J}(p, r) - \mathcal{J}(p, r)(q-1)$. We obtain a stratification

$$\mathcal{V}_{s-1}(n, m, r)(p) = \coprod_{q=0,\ldots,\ell} S_q(s, m, p),$$

where we have that $S_0(s, m, p)$ is equal to $\mathcal{V}_{s-1}(n, m, r)(p-1)$ and where we have that S_q is isomorphic to $\mathcal{V}_s(n, m-p, r) \times \mathcal{J}(p, r)(q-1)$ for $q \geq 1$. Lemma 3.5 tells us that this is in fact an L-stratification. Moreover,

$$\mathcal{V}_{s-1}(n, m, r)(p) = \coprod_{q=0,\ldots,\ell} S_q(s, m, p) \xrightarrow{\iota} \mathcal{V}_{s-1}(n, m+1, r)(p) = \coprod_{q=0,\ldots,\ell} S_q(s, m+1, p)$$

is an L-stratified map.

A first result of this L-stratification is that

$$H_t(\mathcal{V}_{s-1}(n, m+1, r)(m)) \to H_t(\mathcal{V}_{s-1}(n, m+1, r))$$

are isomorphisms for all $t \leq \frac{m}{2}$. Indeed, as $\dim_{\mathbb{C}} \mathcal{J}(m+1, r) = (2r-1)(m+1)$ and $\dim_{\mathbb{C}} V_{s-1}(n, m+1, r) = (2r)(m+1)$, this is a consequence from Lemma 3.7. Because of our assumption, this also implies that

$$H_t(\iota) : H_t(\mathcal{V}_{s-1}(n, m, r)(m)) \to H_t(\mathcal{V}_{s-1}(n, m+1, r)(m))$$

are isomorphisms for all $t \leq \frac{m}{2}$.

Now we claim that

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(\mathrm{p})) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r})(\mathrm{p}))$$

are isomorphisms for all $\mathrm{t} \leq \frac{\mathrm{m}}{2}$ if the same is true for

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(\mathrm{p}+1)) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r})(\mathrm{p}+1)).$$

To show this claim, we look again at the L-stratified map

$$\begin{array}{c} \mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(\mathrm{p}+1) = \coprod_{\mathrm{q}=0,\ldots,\ell} \mathrm{S_q}(\mathrm{s},\mathrm{m},\mathrm{p}+1) \\ \downarrow\iota \\ \mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r})(\mathrm{p}+1) = \coprod_{\mathrm{q}=0,\ldots,\ell} \mathrm{S_q}(\mathrm{s},\mathrm{m}+1,\mathrm{p}+1) \end{array}$$

and verify the following facts:

(i) Since

$$\begin{array}{rcl} \dim_{\mathbb{R}}\mathcal{J}_{\mathrm{q}}(\mathrm{p}+1,\mathrm{r}) & \leq & 4\mathrm{rp}+4\mathrm{r}-2\mathrm{p}-2\mathrm{q} \\ \dim_{\mathbb{R}}\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}-\mathrm{p}-1,\mathrm{r}) & = & 4\mathrm{rm}-4\mathrm{rp}-4\mathrm{r} \\ \dim_{\mathbb{R}}\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(\mathrm{p}+1) & = & 4\mathrm{rm}, \end{array}$$

we have for $\mathrm{q} \geq 1$ the codimension of $\mathrm{S_q}(\mathrm{s},\mathrm{m},\mathrm{p}+1)$ in $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})(\mathrm{p}+1)$ as equal to the codimension of $\mathrm{S_q}(\mathrm{s},\mathrm{m}+1,\mathrm{p}+1)$ in $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r})(\mathrm{p}+1)$ and as greater than or equal to $2\mathrm{p}+2\mathrm{q}$;

(ii) Because of our assumption, the homomorphisms

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}-\mathrm{p}-1,\mathrm{r})) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}-\mathrm{p},\mathrm{r}))$$

are isomorphisms for $\mathrm{t} \leq \frac{\mathrm{m}-\mathrm{p}-1}{2} - 1$. By Lemma 3.10, the homomorphisms

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathrm{S_q}(\mathrm{s},\mathrm{m},\mathrm{p}+1)) \to \mathrm{H_t}(\mathrm{S_q}(\mathrm{s},\mathrm{m}+1,\mathrm{p}+1))$$

are also isomorphisms for $\mathrm{t} \leq \frac{\mathrm{m}-\mathrm{p}-1}{2} - 1$ and $\mathrm{q} \geq 1$;

Combining (i) and (ii), Lemma 3.9 implies that

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathrm{S_0}(\mathrm{s},\mathrm{m},\mathrm{p}+1)) \to \mathrm{H_t}(\mathrm{S_0}(\mathrm{s},\mathrm{m}+1,\mathrm{p}+1))$$

are isomorphisms for $t \le \frac{m}{2} - 1$, which is our above claim. By decreasing induction on p we obtain that

$$H_t(\iota) : H_t(\mathcal{V}_s(n, m, r)) \to H_t(\mathcal{V}_s(n, m+1, r))$$

are isomorphisms for all $t \le \frac{m}{2}$. □

Let $\mathcal{E}(m, r)$ be the space of framed exceptional jumps as in Section 2. We define $\mathcal{E}(m, r)(0)$ to be the open subspace of points (V, α), where V is trivial along one of the two exceptional lines E_0 and E_1. $\mathcal{E}(m, r)(0)$ consists of two components, both isomorphic to $\mathcal{J}(m, r)$. For $i \ge 1$, we define $\mathcal{E}(m, r)(i)$ as the regular locus of the variety $\mathcal{E}(m, r) - \mathcal{E}(m, r)(i-1)$. Thus, we have constructed a finite stratification $\mathcal{E}(m, r) = \coprod_{i \ge 0} \mathcal{E}(m, r)(i)$ of $\mathcal{E}(m, r)$. For $i \ge 1$, the strata $\mathcal{E}(m, r)(i)$ are smooth manifolds. As the dimension of $\mathcal{E}(m, r)$ equals $(2r-1)m$, we have the complex dimension of $\mathcal{E}(m, r)(i)$ as less than or equal to $(2r-1)m - i$. A detailed description of these strata can be obtained from the monadic description of $\mathcal{E}(m, r)$ in Section 2, but will be not used here.

We recall that $\pi : S \to \mathbb{P}^1$ is the blowing up of a ruled surface such that the exceptional fibres of π are precisely $F_1, \dots, F_n$. We fix an integer $n \ge s \ge 1$ and describe F_s as the union of two (-1)-curves E_0 and E_1.

We let $\mathcal{V}_{s-1}(n, m, r, \ell)$ be the subspace of $\mathcal{V}_{s-1}(n, m, r)$ of all points (V, α), where either V has jumping order at F_s less than or equal to ℓ, or where V is trivial along at least one of the exceptional lines E_0 and E_1. We obtain a stratification of the space $\mathcal{V}_{s-1}(n, m, r, \ell)$ into strata $\mathcal{V}_{s-1}(n, m, r, \ell)(i)$ as follows:

- $\mathcal{V}_{s-1}(n, m, r, \ell)(0)$ is the space of all points (V, α) in $\mathcal{V}_{s-1}(n, m, r, \ell)$, where either V is trivial along E_0 or along E_1, or where V has a jump of order less than ℓ at F_s, i.e $\mathcal{V}_{s-1}(n, m, r, \ell)(0) = \mathcal{V}_{s-1}(n, m, r, \ell - 1)$.

- For $i \ge 1$, $\mathcal{V}_{s-1}(n, m, r, \ell)(i) = \mathcal{V}_s(n, m-\ell, r) \times \mathcal{E}(\ell, r)(i)$, where $((V, \alpha), (V', \alpha'))$ corresponds to the point in $\mathcal{V}_{s-1}(n, m, r)$ obtained by adding the framed jump (V', α') to the framed vector sheaf (V, α) at the fibre F_s.

Lemma 3.12 *Assume that*

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_\mathrm{s}(\mathrm{n}, \mathrm{m}, \mathrm{r})) \to \mathrm{H_t}(\mathcal{V}_\mathrm{s}(\mathrm{n}, \mathrm{m}+1, \mathrm{r}))$$

and

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}-1, \mathrm{m}, \mathrm{r})) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}-1, \mathrm{m}+1, \mathrm{r}))$$

are isomorphisms for fixed $\mathrm{n} \geq \mathrm{s} \geq 1$*,* $\mathrm{r} \geq 2$*, and for all* $\mathrm{m} \geq 1$ *and all* $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$*. Then*

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}, \mathrm{m}, \mathrm{r}, 0)) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}, \mathrm{m}+1, \mathrm{r}, 0))$$

are isomorphisms for all $\mathrm{m} \geq 1$ *and all* $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$*.*

Proof. By the definition of the stabilisation map, ι indeed maps $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}, \mathrm{m}, \mathrm{r}, 0)$ to $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}, \mathrm{m}+1, \mathrm{r}, 0)$. The space $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}, \mathrm{m}, \mathrm{r}, 0)$ is the union of the two open subspaces $\mathrm{U_0(m)}$ and $\mathrm{U_1(m)}$, where $\mathrm{U_i(m)}$ contains all points (V, α) with V trivial along $\mathrm{E_i}$. Corresponding to the contraction of $\mathrm{E_i}$, $\mathrm{U_i(m)}$ is isomorphic to $\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}-1, \mathrm{m}, \mathrm{r})$. The intersection $\mathrm{U_{01}(m)} = \mathrm{U_0(m)} \cap \mathrm{U_1(m)}$ is just the space $\mathcal{V}_\mathrm{s}(\mathrm{n}, \mathrm{m}, \mathrm{r})$. Due to our arrangements, we may assume that ι maps $\mathrm{U_i(m)}$ to $\mathrm{U_i(m+1)}$ and $\mathrm{U_{01}(m)}$ to $\mathrm{U_{01}(m+1)}$.

We have the Mayer-Vietoris sequence:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{H_p}(\mathrm{U_0(m)}, \mathbb{Z}) \oplus \mathrm{H_p}(\mathrm{U_1(m)}) & \to & \mathrm{H_p}(\mathrm{U_0(m+1)}, \mathbb{Z}) \oplus \mathrm{H_p}(\mathrm{U_1(m+1)}) \\
\downarrow & & \downarrow \\
\mathrm{H_p}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}, \mathrm{m}, \mathrm{r}, 0)) & \to & \mathrm{H_p}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}, \mathrm{m}+1, \mathrm{r}, 0)) \\
\downarrow & & \downarrow \\
\mathrm{H_{p-1}}(\mathrm{U_{01}(m)}) & \to & \mathrm{H_{p-1}}(\mathrm{U_{01}(m+1)}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

and by our assumptions and a diagram chase, we have proved our claim. □

Lemma 3.13 *Let $\ell \geq 1$ be a fixed integer, and assume that*

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m},\mathrm{r})) \to \mathrm{H_t}(\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m}+1,\mathrm{r})),$$

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}-1,\mathrm{m},\mathrm{r})) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n}-1,\mathrm{m}+1,\mathrm{r}))$$

and

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\ell-1)) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r},\ell-1))$$

are isomorphisms for fixed $\mathrm{n} \geq \mathrm{s} \geq 1$, $\mathrm{r} \geq 2$, *and for all* $\mathrm{m} \geq 1$ *and all* $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$. *The homomorphisms*

$$\mathrm{H_t}(\iota) : \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\ell)) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r},\ell)).$$

are isomorphisms for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$.

Proof. Because of Lemma 3.5, we have

$$\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\ell) = \coprod_{\mathrm{i}\geq 0} \mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\ell)(\mathrm{i})$$

as an L-stratification. Due to its definition, the stabilisation map ι restricts to an L-stratified map:

$$\begin{array}{c}
\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\ell) \\
\downarrow = \\
\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\ell-1) \sqcup \coprod_{\mathrm{i}\geq 1} \mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m}-\ell,\mathrm{r}) \times \mathcal{E}(\ell,\mathrm{r})(\mathrm{i}) \\
\downarrow \iota \\
\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r},\ell-1) \sqcup \coprod_{\mathrm{i}\geq 1} \mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m}-\ell+1,\mathrm{r}) \times \mathcal{E}(\ell,\mathrm{r})(\mathrm{i}) \\
\downarrow = \\
\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r},\ell)
\end{array}$$

By assumption,

$$\mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\ell-1)) \to \mathrm{H_t}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r},\ell-1))$$

are isomorphisms for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$. Because of $\ell \geq 1$, we also have that the homomorphisms

$$\mathrm{H_t}(\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m}-\ell,\mathrm{r})) \to \mathrm{H_t}(\mathcal{V}_\mathrm{s}(\mathrm{n},\mathrm{m}-\ell+1,\mathrm{r}))$$

are isomorphisms for all $t \leq \frac{m-\ell}{2}$, and therefore Lemma 3.10 implies that

$$H_t(\mathcal{V}_s(n, m-\ell, r) \times \mathcal{E}(\ell, r)(i)) \to H_t(\mathcal{V}_s(n, m-\ell+1, r) \times \mathcal{E}(\ell, r)(i))$$

are isomorphisms for all $t \leq \frac{m-\ell}{2}$.

Since

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{V}_{s-1}(n, m, r, \ell) &= 2rm; \\ \dim_{\mathbb{C}} \mathcal{V}_s(n, m-\ell, r) &= 2r(m-\ell); \\ \dim_{\mathbb{C}} \mathcal{E}(\ell, r)(i) &\leq (2r-1)\ell - i; \end{aligned}$$

we obtain $2(\ell + i)$ as a lower bound for the real codimension d_i of $\mathcal{V}_{s-1}(n, m, r, \ell)(i)$ in $\mathcal{V}_{s-1}(n, m, r, \ell)$, $i \geq 1$. Moreover, as d_i does not depend on m, it equals also the real codimension of $\mathcal{V}_{s-1}(n, m+1, r, \ell)(i)$ in $\mathcal{V}_{s-1}(n, m+1, r, \ell)$. We infer that

$$H_t(\mathcal{V}_{s-1}(n, m, r, \ell)(i)) \to H_t(\mathcal{V}_{s-1}(n, m+1, r, \ell)(i))$$

are isomorphisms for all $t \leq \frac{m}{2} + i - d_i$ for all $i \geq 0$. We see that all conditions of Lemma 3.8 are fulfilled and obtain the desired result. □

Proof of Theorem 3.4. The only remaining tasks are some evident inductions: First, we begin with induction on n, the number of blown up points. The start of the induction at the classical situation $n = 0$ is provided by [4] and [40]. The induction step itself is a decreasing induction on $s = n, \ldots, 0$, which shows that the statement holds for

$$\iota : \mathcal{V}_{s-1}(n, m, r) \to \mathcal{V}_{s-1}(n, m+1, r)$$

if it holds for

$$\iota : \mathcal{V}_s(n, m, r) \to \mathcal{V}_s(n, m+1, r).$$

The start of this induction at $s = n$ is given by Proposition 3.11, and the induction step is proved by induction on $\ell = 0, \ldots, m$: if the statement holds for

$$\iota : \mathcal{V}_{s-1}(n, m, r, \ell - 1) \to \mathcal{V}_{s-1}(n, m+1, r, \ell - 1),$$

then it is also true for

$$\iota : \mathcal{V}_{s-1}(n, m, r, \ell) \to \mathcal{V}_{s-1}(n, m+1, r, \ell).$$

The necessary ingredients for this induction are provided by the Propositions 3.14 and 3.15. The induction stops at $\ell = \mathrm{m}$ because of

$$\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r},\mathrm{m}) = \mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m},\mathrm{r})$$

and because of

$$\mathrm{H}_{\mathrm{t}}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r},\mathrm{m})) \cong \mathrm{H}_{\mathrm{t}}(\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r}))$$

for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$. The last statement is an easy consequence from Lemma 3.7 applied to the L-stratification

$$\mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r}) = \coprod_{\mathrm{i}\geq 0} \mathcal{V}_{\mathrm{s}-1}(\mathrm{n},\mathrm{m}+1,\mathrm{r},\mathrm{m}+1)(\mathrm{i}). \ \square$$

3.4 Homotopical stability

The strong part of the results in [4] and [40] is, of course, that the stabilisation map $\iota : \mathcal{V}(0,\mathrm{m},\mathrm{r}) \to \mathcal{V}(0,\mathrm{m}+1,\mathrm{r})$ induces isomorphisms on the homotopy groups $\pi_{\mathrm{t}}\mathcal{V}(0,\mathrm{m},\mathrm{r}) \to \pi_{\mathrm{t}}\mathcal{V}(0,\mathrm{m}+1,\mathrm{r})$ for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 2$. To obtain an analogous homotopy statement for the stabilisation $\iota : \mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r}) \to \mathcal{V}(\mathrm{n},\mathrm{m}+1,\mathrm{r})$, we first determine the fundamental group of $\mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r})$.

Theorem 3.14 *The fundamental group of* $\mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r})$ *does not depend on* n *and* m. *To be precise,*

$$\pi_1(\mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r})) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } \mathrm{r} = 2, \\ 0 & \text{for } \mathrm{r} > 2. \end{cases}$$

Proof. The result is known in the classical case of $\mathrm{n} = 0$ by [18], Theorem 3.13, for $\mathrm{r} = 2$, and by [40], Theorem 4.14, for $\mathrm{r} > 2$. The ruled surface considered is thereby the first Hirzebruch surface, of which our S is a blowing up with exceptional divisor E.

We define $\mathcal{S}_0$ as the open dense submanifold in $\mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r})$ of all points (V,α), where V is trivial along E and has m jumping fibres of order 1. We define further $\mathcal{S}_1$ to be the locally closed submanifold of points (V,α), where V is non-trivial along E and has m jumping fibres of order 1, and moreover $\mathcal{S}_2$ as the locally closed submanifold of points (V, α), where V is trivial along E and has $\mathrm{m} - 2$ jumping fibres of order 1 and one of order 2. We have $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ as an open dense submanifold of $\mathcal{V}(\mathrm{n}, \mathrm{m}, \mathrm{r})$ with a complement of real codimension greater than or equal to 4. Therefore, $\pi_1(\mathcal{V}(\mathrm{n}, \mathrm{m}, \mathrm{r})) = \pi_1(\mathcal{S})$.

The submanifolds $\mathcal{S}_1$ and $\mathcal{S}_2$ have both real codimension 2 in $\mathcal{S}$. Hence, representative loops of classes in $\pi_1(\mathcal{S})$ may be chosen to lie in $\mathcal{S}_0$. As Y. Tian shows explicitly in his proof of [40, Theorem 4.14], and as it has been shown by J. Hurtubise in the case of $\mathrm{r} = 2$ before ([18, Theorem 3.13]), any loop in $\mathcal{S}_0$ is homotopic in $\mathcal{S}_0 \cup \mathcal{S}_2$ to a loop in a general fibre of $\Delta : \mathcal{S}_0 \to \mathrm{Sym}^{\mathrm{m}}(\mathbb{C})$. Hence, we may represent any class in $\pi_1(\mathcal{S})$ by a loop in such a fibre of $\Delta|_{\mathcal{S}_0}$, as well. Consequently, $\pi_1(\mathcal{S})$ is a quotient of $\pi_1(\mathcal{J}(1, \mathrm{r})^{\mathrm{m}})$ by a subgroup.

Since $\pi_1(\mathcal{J}(\mathrm{m}, \mathrm{r})) = 0$ for $\mathrm{r} > 2$, by [40, Lemma 4.12], it remains only to examine the case $\mathrm{r} = 2$. There we have $\mathcal{J}(1, 2)$ homotopy equivalent to SO(3) by [18, Proposition 3.10], and therefore $\pi_1(\mathcal{J}(\mathrm{m}, \mathrm{r})) \cong \mathbb{Z}/2\mathbb{Z}$. We denote the generators of $\pi_1(\mathcal{J}(1, 2)^{\mathrm{m}}) \cong (\mathbb{Z}/2\mathbb{Z})^{\mathrm{m}}$ by $(0, \ldots, 0, 1, 0, \ldots, 0)$. It is not difficult to see, that $(1, 0, \ldots, 0)$ is not contractible in $\mathcal{S}$. On the other side, monodromies in $\mathcal{S}_0$ interchange the generators (cf. [40, Remark 4.21]), and we obtain $\pi_1(\mathcal{V}(\mathrm{n}, \mathrm{m}, 2)) \cong \mathbb{Z}/2\mathbb{Z}$. □

Theorem 3.15 *The stabilisation map* $\iota : \mathcal{V}(\mathrm{n}, \mathrm{m}, \mathrm{r}) \to \mathcal{V}(\mathrm{n}, \mathrm{m}+1, \mathrm{r})$ *induces isomorphisms on the homotopy groups*

$$\pi_{\mathrm{t}}(\iota) : \pi_{\mathrm{t}}(\mathcal{V}(\mathrm{n}, \mathrm{m}, \mathrm{r})) \cong \pi_{\mathrm{t}}(\mathcal{V}(\mathrm{n}, \mathrm{m}+1, \mathrm{r}))$$

for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 2$.

Proof. In the case of $\mathrm{r} > 2$, the fundamental group of $\mathcal{V}(\mathrm{n}, \mathrm{m}, \mathrm{r})$ vanishes by the previous theorem, and our claim holds as a direct consequence of Theorem 3.4 by the Whitehead Theorem [5, Theorem VII.11.2].

For the case of $\mathrm{r} = 2$, we consider the diagram

$$\begin{array}{ccc} \tilde{\mathcal{V}}(\mathrm{n},\mathrm{m},\mathrm{r}) & \xrightarrow{\tilde{\iota}} & \tilde{\mathcal{V}}(\mathrm{n},\mathrm{m}+1,\mathrm{r}) \\ \downarrow & & \downarrow \\ \mathcal{V}(\mathrm{n},\mathrm{m},\mathrm{r}) & \xrightarrow{\iota} & \mathcal{V}(\mathrm{n},\mathrm{m}+1,\mathrm{r}) \end{array}$$

of the universal two-fold coverings. We note that the inverse images of an L-stratification on $\mathcal{V}(\mathrm{n},\mathrm{m},2)$ induce an L-stratification on $\tilde{\mathcal{V}}(\mathrm{n},\mathrm{m},2)$ and make the covering to an L-stratified map. Moreover, this is an L-stratified map preserving the codimension of the strata.

In the following, we obtain the two-fold coverings of the strata and submanifolds under discussion by embedding them into some $\mathcal{V}(\mathrm{n},\mathrm{m},2)$ and restricting the associated covering. Because of the universal property, it does not matter in which of the possible $\mathcal{V}(\mathrm{n},\mathrm{m},2)$ we have embedded.

From [4] we know that

$$\mathrm{H}_{\mathrm{t}}(\tilde{\mathcal{V}}(0,\mathrm{m},2)) \to \mathrm{H}_{\mathrm{t}}(\mathcal{V}(0,\mathrm{m},2))$$

are isomorphisms for all $\mathrm{t} < \mathrm{m}$. If we recall the L-stratifications constructed in the proof of Proposition 3.11, we obtain by Lemma 3.9 that

$$\mathrm{H}_{\mathrm{t}}(\tilde{\mathcal{V}}_{\mathrm{s}}(0,\mathrm{m},2)) \to \mathrm{H}_{\mathrm{t}}(\mathcal{V}_{\mathrm{s}}(0,\mathrm{m},2))$$

are isomorphisms for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$. We also learn that

$$\mathrm{H}_{\mathrm{t}}(\tilde{\mathcal{V}}_{\mathrm{s}}(0,\mathrm{m},2)) \to \mathrm{H}_{\mathrm{t}}(\tilde{\mathcal{V}}_{\mathrm{s}}(0,\mathrm{m}+1,2)).$$

In this way we proceed with Lemma 3.12 and 3.13; i.e., we replace in the assumptions and assertions the conditions

> $\mathrm{H}_{\mathrm{t}}(\iota) : \mathrm{H}_{\mathrm{t}}(\mathrm{S}_{\mathrm{m}}) \to \mathrm{H}_{\mathrm{t}}(\mathrm{S}_{\mathrm{m}+1})$
> is an isomorphism for certain strata S_{m} and certain t

by

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{t}}(\tilde{\mathrm{S}}_{\mathrm{m}}) & \xrightarrow{\mathrm{H}_{\mathrm{t}}(\tilde{\iota})} & \mathrm{H}_{\mathrm{t}}(\tilde{\mathrm{S}}_{\mathrm{m}+1}) \\ \downarrow & & \downarrow \\ \mathrm{H}_{\mathrm{t}}(\mathrm{S}_{\mathrm{m}}) & \xrightarrow{\mathrm{H}_{\mathrm{t}}(\iota)} & \mathrm{H}_{\mathrm{t}}(\mathrm{S}_{\mathrm{m}+1}) \end{array}$$

> is a diagram of isomorphisms for those S_{m} and t.

The proofs of both lemmas are easily generalised for the statements so obtained, and the final proof of Theorem 3.4 can be adapted as well. We obtain that

$$\begin{array}{ccc}
\mathrm{H}_{\mathrm{t}}(\tilde{\mathcal{V}}(\mathrm{n},\mathrm{m},2)) & \xrightarrow{\mathrm{H}_{\mathrm{t}}(\tilde{\iota})} & \mathrm{H}_{\mathrm{t}}(\tilde{\mathcal{V}}(\mathrm{n},\mathrm{m}+1,2)) \\
\downarrow & & \downarrow \\
\mathrm{H}_{\mathrm{t}}(\mathcal{V}(\mathrm{n},\mathrm{m},2)) & \xrightarrow{\mathrm{H}_{\mathrm{t}}(\iota)} & \mathrm{H}_{\mathrm{t}}(\mathcal{V}(\mathrm{n},\mathrm{m}+1,\mathrm{r}))
\end{array}$$

is a diagram of isomorphisms for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 1$.

It is quite obvious that

$$\pi_{\mathrm{t}}(\iota) : \pi_{\mathrm{t}}(\mathcal{V}(\mathrm{n},\mathrm{m},2)) \longrightarrow \pi_{\mathrm{t}}(\mathcal{V}(\mathrm{n},\mathrm{m}+1,\mathrm{r}))$$

is an isomorphism. Indeed, we have seen in the proof of Theorem 3.14 that both fundamental groups are generated by the fundamental group of a general fibre along Δ and $\pi_{\mathrm{t}}(\iota)$ is of the form

$$\pi_{\mathrm{t}}(\mathcal{J}(1,2))^{\mathrm{m}}/\mathrm{Sym}_{\mathrm{m}} \longrightarrow \pi_{\mathrm{t}}(\mathcal{J}(1,2))^{\mathrm{m}+1}/\mathrm{Sym}_{\mathrm{m}+1},$$

where the symmetric groups act by permutation. Since ι just adds a new jump of order one, the map is moreover induced by

$$\pi_{\mathrm{t}}(\mathcal{J}(1,2))^{\mathrm{m}} \longrightarrow \pi_{\mathrm{t}}(\mathcal{J}(1,2))^{\mathrm{m}} \times \{0\}$$

and therefore an isomorphism.

The universal two-fold covering is in particular a fibration, where the higher homotopy groups π_{t} of the fibre vanish for $\mathrm{t} \geq 2$. From the exact homotopy sequence of a fibration ([5, Theorem VII.6.7]) we infer that

$$\pi_{\mathrm{t}}(\tilde{\mathcal{V}}(\mathrm{n},\mathrm{m},2)) \longrightarrow \pi_{\mathrm{t}}(\mathcal{V}(\mathrm{n},\mathrm{m},2))$$

is an isomorphism for all $\mathrm{t} \geq 2$.

As universal coverings are simply connected, Whitehead's Theorem and the above homology considerations yield

$$\pi_{\mathrm{t}}(\tilde{\mathcal{V}}(\mathrm{n},\mathrm{m},2)) \cong \pi_{\mathrm{t}}(\tilde{\mathcal{V}}(\mathrm{n},\mathrm{m}+1,2))$$

for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 2$, and the naturality of the homotopy functor implies

$$\pi_{\mathrm{t}}(\mathcal{V}(\mathrm{n},\mathrm{m},2)) \cong \pi_{\mathrm{t}}(\mathcal{V}(\mathrm{n},\mathrm{m}+1,2))$$

for all $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 2$. □

Thus we have proved, in generalisation of [4] and [40], the following Atiyah-Jones type result.

Theorem 3.16 *Let* M *be a self-dual oriented Riemannian 4-dimensional manifold with twistor fibration* $\mathrm{P} \to \mathrm{M}$, *such that P contains a surface of degree 1, which in turn contains the twistor fibre over a point* $\mathrm{x} \in \mathrm{M}$. *For* $\mathrm{r} \geq 2$, *there is a stabilisation map* $\iota : \mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r}) \to \mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}+1,\mathrm{r})$ *between the moduli spaces of based* $\mathrm{SU(r)}$*-instantons of charge* m *and charge* $\mathrm{m}+1$ *on* (M,x), *such that the induced homomorphisms* $\pi_{\mathrm{t}}(\iota) : \pi_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m},\mathrm{r})) \to \pi_{\mathrm{t}}(\mathcal{I}ns^{\mathrm{SU(r)}}(\mathrm{m}+1,\mathrm{r}))$ *on the homotopy groups are isomorphisms for all* $\mathrm{t} \leq \frac{\mathrm{m}}{2} - 2$. *In the classical situation* $\mathrm{M} = \mathrm{S}^4$, ι *is homotopy equivalent to the original stabilisation map of Taubes ([39]) and the ones used in [4] and [40].*

3.5 Instantons on blown up ruled surfaces

It is quite obvious that we may apply the techniques of Sections 3.2 and 3.3 to generalise the results of J.C. Hurtubise and R.J. Milgram in [19] in the following way:

Theorem 3.17 *Let* X *be the blowing up of a ruled complex surface* X' *in* n *points, of which no two are situated on the same ruling fibre. For* $\mathcal{M}_{\mathrm{m}}$ *the moduli space of* $\mathrm{SU}(2)$*-instantons of charge* m *on* X, *there is a stabilisation map* $\iota : \mathcal{M}_{\mathrm{m}} \to \mathcal{M}_{\mathrm{m}+1}$ *such that* $\mathrm{H}_{\mathrm{t}}\iota$ *is an isomorphism for all* $0 \leq \mathrm{t} \leq \frac{\mathrm{m}}{2} - \mathrm{c}_1(\mathrm{X})$. *Hereby,* $\mathrm{c}_1(\mathrm{X}) = \mathrm{c}_1(\mathrm{X}')$ *coincides with the* c_1 *explicitly computed in [19]. Moreover, in the case* $\mathrm{X} = \mathrm{X}'$, *the stabilisation map is the same as the one constructed in [19].*

Theorem 3.18 *Let* X *be the blowing up of* $\mathbb{P}^2_{\mathbb{C}}$ *in* n *distinct points. For* $\mathcal{M}_{\mathrm{m}}$ *the moduli space of* $\mathrm{SU}(2)$*-instantons of charge* m *on* X, *there is a stabilisation map*

$\iota : \mathcal{M}_m \to \mathcal{M}_{m+1}$ such that $H_t\iota$ is an isomorphism for all $0 \le t \le \frac{m}{2} - 2$. In the case $X = \mathbb{P}^2_{\mathbb{C}}$, the stabilisation map is the same as the one constructed in [19].

4 Completion of the moduli space

4.1 An embedding into a quotient scheme

We recall now the general setup, where S' was isomorphic to the blowing up of the projective plane over the field k in some zero-dimensional centre and F' was a line on S' not meeting the exceptional locus E. We may think of $S' \to \mathbb{P}^2_k$ as the composition of n monoidal transformations for some $n \ge 1$.

From now on, we consider two smooth rational curves N and F over k and put S to be the blowing up of $N \times F$ in some zero-dimensional centre with exceptional divisor E. We have two flat morphisms $\pi : S \to N$ and $\kappa : S \to F$ induced by the two projections. For P and Q points on N and F, we denote with F_P and N_Q the fibres of π and κ over P and Q, respectively. We set $\eta \in N$ and $\xi \in F$ to be the generic points on our curves.

Moreover, we fix two closed points $P_\infty \in N$ and $Q_\infty \in F$, such that the fibres $F_\infty = F_{P_\infty}$ and $N_\infty = N_{Q_\infty}$ are sections of κ and π. For $\mathcal{F}$ an arbitrary module sheaf on S and two integers p and q, we denote with $\mathcal{F}(p, q)$ the tensor product $\mathcal{F} \otimes \pi^*\mathcal{O}_N(pP_\infty) \otimes \pi^*\mathcal{O}_F(qQ_\infty)$. We note that the canonical divisor on S is of the form $K_S = \mathcal{O}_S(-2, -2)(E')$, where E' is an effective divisor with support on E and self-intersection $(E')^2 = -n$.

$\mathcal{V}(m, r)$ denotes the functor as well as the fine moduli space of framed vector sheaves (V, α) on S, where V is a vector sheaf with rank r, trivial determinant and second Chern class equal to m, which is trivial along $F_\infty \cup N_\infty$, and where the framing α is a fixed isomorphism $V|_{F_\infty \cup N_\infty} \cong \mathcal{O}^r_{F_\infty \cup N_\infty}$.

We think of S as the birational model of S' obtained by blowing up S' in two distinct points on F' and then contracting the strict transform of F'. From there, it is obvious that $\mathcal{V}(m, r)$ is naturally isomorphic to the moduli space of framed vector sheaves $\mathcal{V}ec(\mathrm{S}', \mathrm{F}', \mathcal{O}^{\mathrm{r}}_{\mathrm{F}'}, \mathrm{c}_1 = 0, \mathrm{c}_2 = \mathrm{m})$.

If (V, α) is an element in $\mathcal{V}(\mathrm{m}, \mathrm{r})(\operatorname{Spec} \mathrm{k})$, then $\mathrm{V}(-1, -1)$ has vanishing Euler characteristic along all fibres of π and κ, and moreover vanishing cohomology along general fibres. Due to the determinantal divisor with respect to π and κ, we therefore obtain two morphisms $\Delta : \mathcal{V}(\mathrm{m}, \mathrm{r}) \to \mathrm{Sym}^{\mathrm{m}}\mathrm{N}$ and , $: \mathcal{V}(\mathrm{m}, \mathrm{r}) \to \mathrm{Sym}^{\mathrm{m}}\mathrm{F}$ as in Section 1.3.

In the following, we construct for each point (V, α) in $\mathcal{V}(\mathrm{m}, \mathrm{r})$ monomorphisms $\beta : \mathrm{V} \hookrightarrow \mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(\mathrm{m}, 0)$ and $\gamma : \mathrm{V} \hookrightarrow \mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(0, \mathrm{m})$. As we would like for these maps to be natural, we actually construct them for families.

Lemma 4.1 *Let* T *be an integral algebraic* k*-scheme and* $(\tilde{\mathrm{V}}, \tilde{\alpha})$ *an element in* $\mathcal{V}(\mathrm{m}, \mathrm{r})(\mathrm{T})$*. Then* $\tilde{\alpha}$ *induces monomorphisms*

$$\tilde{\beta} : \tilde{\mathrm{V}} \hookrightarrow \mathcal{O}_{\mathrm{T}} \boxtimes \mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(\mathrm{m}, 0)$$

and

$$\tilde{\gamma} : \tilde{\mathrm{V}} \hookrightarrow \mathcal{O}_{\mathrm{T}} \boxtimes \mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(0, \mathrm{m}),$$

which are canonically determined as elements in the natural sets $\mathrm{Quot}(\mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(\mathrm{m}, 0))(\mathrm{T})$ *and* $\mathrm{Quot}(\mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(0, \mathrm{m}))(\mathrm{T})$ *by the isomorphism class of the framed module* $(\tilde{\mathrm{V}}, \tilde{\alpha})$*.*

Proof. We put $\mathrm{W} = \mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(\mathrm{m}, 0)$ and $\tilde{\mathrm{W}} = \mathcal{O}_{\mathrm{T}} \boxtimes \mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(\mathrm{m}, 0)$. Because of symmetry, it is sufficient to prove the lemma for W.

Let $\tilde{\mathrm{N}} = \mathrm{T} \times \mathrm{N}$, $\tilde{\mathrm{N}}_\infty = \mathrm{T} \times \mathrm{N}_\infty$ and $\tilde{\pi} = \mathrm{id}_{\mathrm{T}} \times \pi$. If we fix coordinates on N, we obtain a fixed isomorphism

$$\mathcal{O}^{\mathrm{r}}_{\mathrm{T} \times \mathrm{S}}(\mathrm{D}) \cong \tilde{\mathrm{W}}$$

for $\mathrm{D} \subset \tilde{\mathrm{N}}$ any family of effective divisors of order m in N. The short exact sequence

$$0 \to \tilde{\mathrm{V}}^\vee(-\tilde{\mathrm{N}}_\infty) \to \tilde{\mathrm{V}}^\vee \to \tilde{\mathrm{V}}^\vee|_{\tilde{\mathrm{N}}_\infty} \to 0$$

yields the exact sequence

$$0 \to \tilde{\pi}_*\tilde{\mathrm{V}}^\vee \to \tilde{\mathrm{V}}^\vee|_{\tilde{\mathrm{N}}_\infty} \to \mathrm{R}^1\tilde{\pi}_*\tilde{\mathrm{V}}^\vee(-\tilde{\mathrm{N}}_\infty) \to \mathrm{R}^1\tilde{\pi}_*\tilde{\mathrm{V}}^\vee \to 0,$$

where we have identified $\tilde{\mathrm{N}}$ and $\tilde{\mathrm{N}}_\infty$ via $\tilde{\pi}$. The kernel $\mathcal{K}$ of $\mathrm{R}^1\tilde{\pi}_*\tilde{\mathrm{V}}^\vee(-\tilde{\mathrm{N}}_\infty) \to \mathrm{R}^1\tilde{\pi}_*\tilde{\mathrm{V}}^\vee$ is an $\mathcal{O}_{\mathrm{D}}$-module for D some family of effective divisors of order m appearing as the determinantal divisor associated to $\tilde{\mathrm{V}}^\vee(-\tilde{\mathrm{N}}_\infty)$.

From the exact sequence

$$0 \to \tilde{\pi}^*\tilde{\pi}_*\tilde{\mathrm{V}}^\vee \to \tilde{\pi}^*\tilde{\mathrm{V}}^\vee|_{\tilde{\mathrm{N}}_\infty} \to \tilde{\pi}^*\mathcal{K} \to 0$$

we thus obtain a factorization $\tilde{\pi}^*\tilde{\mathrm{V}}^\vee|_{\tilde{\mathrm{N}}}(-\mathrm{D}) \hookrightarrow \tilde{\pi}^*\tilde{\pi}_*\tilde{\mathrm{V}}^\vee$. The framing $\tilde{\alpha}$ induces $\tilde{\pi}^*\tilde{\mathrm{V}}^\vee|_{\tilde{\mathrm{N}}_\infty}(-\mathrm{D}) \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{T}\times\mathrm{S}}(-\mathrm{D}) = \mathrm{W}^\vee$. Moreover, we note that the natural homomorphism $\tilde{\pi}^*\tilde{\pi}_*\tilde{\mathrm{V}}^\vee \to \mathrm{V}^\vee$ is injective, too. Hence, we have a monomorphism

$$\tilde{\mathrm{W}}^\vee \hookrightarrow \tilde{\mathrm{V}}^\vee$$

with a cokernel of pure torsion. By dualization we obtain the desired monomorphism

$$\tilde{\beta} : \tilde{\mathrm{V}} \hookrightarrow \tilde{\mathrm{W}}.$$

Since $\tilde{\mathrm{V}}$ is locally free, the whole construction is compatible with restriction to a fibre over a point $\mathrm{t} \in \mathrm{T}$. In other words, $\tilde{\beta}|_{\{\mathrm{t}\}\times\mathrm{S}}$ is still injective. Therefore, if $\tilde{\mathrm{Q}}$ denotes the cokernel of $\tilde{\beta}$, then the Chern classes of $\tilde{\mathrm{Q}}|_{\{\mathrm{t}\}\times\mathrm{S}}$ are locally constant over T. Hence, by [13], Theorem III.9.9., $\tilde{\mathrm{Q}}$ is flat over T and $\tilde{\beta}$ represents an element in Quot(W)(T).

Since two isomorphisms between $\mathcal{O}^{\mathrm{r}}_{\mathrm{T}\times\mathrm{S}}(\mathrm{D})$ and $\tilde{\mathrm{W}}$ resulting from two different choices of coordinates along N differ by a scalar, $\tilde{\beta}$ is well determined as an element in Quot(W)(T) without fixing any coordinates. Obviously, this element does not depend on the choice of a representative in the isomorphism class of $(\tilde{\mathrm{V}}, \tilde{\alpha})$. $\square$

The functor $\mathcal{V}(\mathrm{m}, \mathrm{r})$ is represented by a smooth algebraic space over k. This space admits an etale covering by integral algebraic k-schemes. Thus, the above natural mappings

$$\mathcal{V}(\mathrm{m}, \mathrm{r})(\mathrm{T}) \to \mathrm{Quot}(\mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(\mathrm{m}, 0))(\mathrm{T})$$

and

$$\mathcal{V}(\mathrm{m}, \mathrm{r})(\mathrm{T}) \to \mathrm{Quot}(\mathcal{O}^{\mathrm{r}}_{\mathrm{S}}(0, \mathrm{m}))(\mathrm{T})$$

for integral algebraic k-schemes T extend to natural transformations

$$\mathcal{V}(\mathrm{m},\mathrm{r}) \to \mathrm{Quot}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0))$$

and

$$\mathcal{V}(\mathrm{m},\mathrm{r}) \to \mathrm{Quot}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(0,\mathrm{m})).$$

Moreover, if we have fixed an ample divisor H on S, then we have natural transformations from $\mathcal{V}(\mathrm{m},\mathrm{r})$ to the projective quotient schemes $\mathrm{Quot}^{\mathrm{f}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0))$ and $\mathrm{Quot}^{\mathrm{g}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(0,\mathrm{m}))$, where f and g are certain Hilbert polynomials.

Theorem 4.2 *The natural transformations*

$$\mathcal{V}(\mathrm{m},\mathrm{r}) \to \mathrm{Quot}^{\mathrm{f}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0))$$

and

$$\mathcal{V}(\mathrm{m},\mathrm{r}) \to \mathrm{Quot}^{\mathrm{g}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(0,\mathrm{m}))$$

are open embeddings.

Proof. Note that all three functors are represented by fine moduli spaces. The morphisms due to Lemma 4.1 are obviously injective, and we will show the bijectivity of the tangent maps. Again, we put $\mathrm{W} = \mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0)$ and, because of symmetry, it is enough to prove the Theorem for W.

Let (V,α) be a point in $\mathcal{V}(\mathrm{m},\mathrm{r})$. Since for a free vector sheaf along a projective line a trivialisation is already given by a framing in one point, we may consider α as restricted to N_∞, as long as we do not omit the condition that $\mathrm{V}|_{\mathrm{F}_\infty}$ is free. Thus, the tangent space of (V,α) in $\mathcal{V}(\mathrm{m},\mathrm{r})$ is naturally identified with $\mathrm{Ext}^1(\mathrm{V},\mathrm{V}(-\mathrm{N}_\infty))$ (cf. [33]). By the proof of [33], Theorem 1.2, there is a one-to-one correspondence between tangent vectors $\mathrm{e} \in \mathrm{Ext}^1(\mathrm{V},\mathrm{V}(-\mathrm{N}_\infty))$ and commutative diagrams with exact lines:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{V} & \longrightarrow & \tilde{\mathrm{V}} & \longrightarrow & \mathrm{V} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{V}|_{\mathrm{N}_\infty} & \longrightarrow & \tilde{\mathrm{V}}_{\mathrm{N}_\infty} & \longrightarrow & \mathrm{V}|_{\mathrm{N}_\infty} & \to & 0 \\
 & & \downarrow\alpha & & \downarrow\tilde{\alpha} & & \downarrow\alpha & & \\
0 & \to & \mathcal{O}_{\mathrm{N}_\infty}^{\mathrm{r}} & \longrightarrow & \mathcal{O}_{\mathrm{N}_\infty}^{\mathrm{r}} \oplus \mathcal{O}_{\mathrm{N}_\infty}^{\mathrm{r}} & \longrightarrow & \mathcal{O}_{\mathrm{N}_\infty}^{\mathrm{r}} & \to & 0
\end{array}$$

On the other side, for $0 \to \mathrm{V} \xrightarrow{\beta} \mathrm{W} \longrightarrow \mathrm{Q} \to 0$ a point in the image of $\mathcal{V}(\mathrm{m}, \mathrm{r})$ in $\mathrm{Quot}^{\mathrm{f}}\mathrm{W}$, the tangent space is naturally identified with $\mathrm{Hom}(\mathrm{V}, \mathrm{Q})$. From the proof of this fact in [12], we learned that there is a one-to-one correspondence between tangent vectors $\varphi \in \mathrm{Hom}(\mathrm{V}, \mathrm{Q})$ and commutative diagrams with exact lines and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{V} & \xrightarrow{\beta} & \mathrm{W} & \longrightarrow & \mathrm{Q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{\mathrm{V}} & \xrightarrow{\tilde{\beta}} & \mathrm{W} \oplus \mathrm{W} & \longrightarrow & \tilde{\mathrm{Q}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{V} & \xrightarrow{\beta} & \mathrm{W} & \longrightarrow & \mathrm{Q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In terms of these diagrams, the tangent map

$$\mathrm{Ext}^1(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\infty)) \to \mathrm{Hom}(\mathrm{V}, \mathrm{Q})$$

is described as follows: If U is the maximal neighbourhood on N_∞ such that V is trivial along all fibres over U, then $\tilde{\mathrm{V}}$ is again trivial along all such fibres. In the same way as α induces an isomorphism $\beta' : \mathrm{V}|_{\pi^{-1}\mathrm{U}} \cong \mathcal{O}^{\mathrm{r}}_{\pi^{-1}\mathrm{U}}$, $\tilde{\alpha}$ induces an isomorphism $\tilde{\beta}' : \tilde{\mathrm{V}}|_{\pi^{-1}\mathrm{U}} \cong \mathcal{O}^{\mathrm{r}}_{\pi^{-1}\mathrm{U}} \oplus \mathcal{O}^{\mathrm{r}}_{\pi^{-1}\mathrm{U}}$, and in fact, $\tilde{\beta}' = \beta' \oplus \beta'$. Hence, the pole order of $\tilde{\beta}'$ along the complement of $\pi^{-1}\mathrm{U}$ is also not greater than m, and we obtain an extension $\tilde{\beta} : \tilde{\mathrm{V}} \to \mathrm{W} \oplus \mathrm{W}$ compatible with the β's obtained from Lemma 4.1. Moreover, we have constructed the diagram belonging to the image under the tangent map.

We obtain an inverse to this tangent map by observing that, starting with a diagram belonging to a tangent vector at a point in the image of $\mathcal{V}(\mathrm{m}, \mathrm{r})$, $\tilde{\beta}$ induces a framing of $\tilde{\mathrm{V}}$ along F_η, and hence by restriction a framing in the generic point of N_∞. Since $\tilde{\mathrm{V}}$ is again trivial along N_∞, this framing induces an isomorphism $\tilde{\alpha} : \tilde{\mathrm{V}}|_{\mathrm{N}_\infty} \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{N}_\infty} \oplus \mathcal{O}^{\mathrm{r}}_{\mathrm{N}_\infty}$ compatible with α. $\square$

4.2 Smoothness of the completion

Recall that $\mathcal{V}(\mathrm{m},\mathrm{r})$ is smooth and irreducible. We define $\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})$ and $\mathcal{V}_\Gamma(\mathrm{m},\mathrm{r})$ to be the closures of $\mathcal{V}(\mathrm{m},\mathrm{r})$ in $\mathrm{Quot}^{\mathrm{f}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0))$ and $\mathrm{Quot}^{\mathrm{g}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(0,\mathrm{m}))$, respectively. Of course, the projective varieties obtained in this manner do not depend on the chosen ample divisor H.

Lemma 4.3

(i) Let $\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0) \twoheadrightarrow \mathrm{Q}$ *be a point in* $\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})$. *Then* Q *is concentrated in finitely many fibres of* π.

(ii) Let $\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(0,\mathrm{m}) \twoheadrightarrow \mathrm{Q}$ *be a point in* $\mathcal{V}_\Gamma(\mathrm{m},\mathrm{r})$. *Then* Q *is concentrated in finitely many fibres of* κ.

Proof. Again, because of symmetry, we have only to show *(i)*.

As a cokernel of a monomorphism of a sheaf of rank r into a vector sheaf of rank r, Q is a direct sum of torsion concentrated in points and torsion concentrated on curves. We are to show that with respect to π no horizontal curves may occur. For that, we consider the Hilbert polynomial $\chi(\mathrm{Q}(\ell\mathrm{F}_\infty))$. Since Hilbert polynomials are locally constant and additive in exact sequences, we may compute this Hilbert polynomial as the difference $\chi(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0)(\ell\mathrm{F}_\infty)) - \chi(\mathrm{V}(\ell\mathrm{F}_\infty))$ for (V,α) arbitrarily chosen in $\mathcal{V}(\mathrm{m},\mathrm{r})$. In this way, we obtain $\chi(\mathrm{Q}(\ell\mathrm{F}_\infty)) = \mathrm{rm}+\mathrm{m} = \text{constant}$. But for any horizontal curves of π, the restriction of F_∞ is ample. Hence, if the support of Q would contain horizontal curves, $\chi(\mathrm{Q}(\ell\mathrm{F}))$ would be a polynomial of degree 1. □

Corollary 4.4

(i) Let $\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0) \twoheadrightarrow \mathrm{Q}$ *be a point in* $\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})$. *The kernel* V *of this quotient restricted to* F_η *is free of rank r.*

(ii) Let $\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(0,\mathrm{m}) \twoheadrightarrow \mathrm{Q}$ *be a point in* $\mathcal{V}_\Gamma(\mathrm{m},\mathrm{r})$. *The kernel* V *of this quotient restricted to* N_ξ *is free of rank r.*

Proof. Again, we are to show only *(i)*. Since Q is concentrated in some fibres of π, there is an open neighbourhood U in N, such that $\mathrm{V}|_{\pi^{-1}\mathrm{U}} \hookrightarrow \mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0)|_{\pi^{-1}\mathrm{U}}$ is an isomorphism. And the restriction of $\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0)$ to F_η is free. □

Theorem 4.5 $\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})$ *and* $\mathcal{V}_\Gamma(\mathrm{m},\mathrm{r})$ *are smooth projective varieties of dimension* $2\mathrm{mr}$.

Proof. It is enough to show the statement for $\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})$. This variety is projective, irreducible and of dimension $2\mathrm{mr}$ as closure of $\mathcal{V}(\mathrm{m},\mathrm{r})$ in the projective scheme $\mathrm{Quot}^{\mathrm{f}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0))$. It remains to show the smoothness.

We define $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})$ as the subfunctor of $\mathrm{Quot}^{\mathrm{f}}\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0)$ which associates to an algebraic space T all those elements

$$\mathcal{O}_{\mathrm{T}} \boxtimes \mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0) \twoheadrightarrow \tilde{\mathrm{Q}} \in \mathrm{Quot}^{\mathrm{f}}(\mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0))(\mathrm{T}),$$

where $\tilde{\mathrm{Q}} \otimes \mathrm{k}(\mathrm{t})$ is concentrated in finitely many fibres over $\mathrm{Spec}\,\mathrm{k}(\mathrm{t}) \times \mathrm{N}$ for all points $\mathrm{t} \in \mathrm{T}$. Because of Lemma 4.3, we have $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r}) \hookrightarrow \mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})$. It is enough to show that the deformation functor of all points in $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})(\mathrm{Spec}\,\mathrm{k})$ is formally smooth, which is provided by the following lemma. □

Lemma 4.6 *Let* $\mathrm{W} = \mathcal{O}_{\mathrm{S}}^{\mathrm{r}}(\mathrm{m},0)$ *and consider an element*

$$\mathrm{W} \twoheadrightarrow \mathrm{Q} \in \mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})(\mathrm{Spec}\,\mathrm{k})$$

with kernel $\beta : \mathrm{V} \hookrightarrow \mathrm{W}$ *and an arbitrary fibre* N_ξ *of* κ *isomorphic to* $\mathbb{P}^1_{\mathrm{k}}$.

(i) The functor $\mathrm{Def}_{\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})}(\mathrm{W} \twoheadrightarrow \mathrm{Q})$ *is naturally equivalent to a deformation functor* $\mathrm{Def}(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$ *of framed module sheaves.*

(ii) The resulting functor $\mathrm{Def}(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$ *is formally smooth.*

Proof. Let $(\mathrm{T}, \mathrm{t}_0)$ be a pointed k-scheme and

$$0 \to \tilde{\mathrm{V}} \xrightarrow{\tilde{\beta}} \tilde{\mathrm{W}} \longrightarrow \tilde{\mathrm{Q}} \to 0$$

an element of $\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})(\mathrm{T})$ together with an identification $\tilde{\beta}|_{\mathrm{t}_0\times \mathrm{S}} = \beta$. We define α and $\tilde{\alpha}$ as the composition of β and $\tilde{\beta}$ with the two restrictions $\mathrm{W} \to \mathrm{W}|_{\mathrm{N}_\xi}$ and $\tilde{\mathrm{W}} \to \tilde{\mathrm{W}}|_{\mathrm{T}\times\mathrm{N}_\xi}$, respectively. In this natural way we have obtained a deformation $(\tilde{\mathrm{V}}, \tilde{\alpha} : \tilde{\mathrm{V}} \to \tilde{\mathrm{W}}|_{\mathrm{T}\times\mathrm{N}_\xi})$ of the framed module $(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$.

Conversely, let $(\tilde{\mathrm{V}}, \tilde{\alpha} : \tilde{\mathrm{V}} \to \tilde{\mathrm{W}}|_{\mathrm{T}\times\mathrm{N}_\xi})$ be a deformation over $(\mathrm{T}, \mathrm{t}_0)$ of such a framed module $(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$ obtained as above. By Corollary 4.4, we know that after restricting T to a suitable open neighbourhood of t_0, the restriction of $\tilde{\mathrm{V}}$ to $\mathrm{T} \times \mathrm{F}_\eta$ is free of rank r. The determinantal divisor associated to the module $\mathrm{R}^1\tilde{\pi}_*\tilde{\mathrm{V}}(-\mathrm{T} \times \mathrm{N}_\xi)$ is a family of effective divisors of order m on N. For U the complement of the support of $\mathrm{R}^1\tilde{\pi}_*\tilde{\mathrm{V}}(-\mathrm{T} \times \mathrm{N}_\xi)$, we have $\tilde{\pi}^*\tilde{\mathrm{V}}|_{\mathrm{U}} = \tilde{\mathrm{V}}|_{\tilde{\pi}^{-1}\mathrm{U}}$, and $\tilde{\alpha}^*$ induces a monomorphism

$$\tilde{\beta}' : \tilde{\mathrm{V}}|_{\tilde{\pi}^{-1}\mathrm{U}} \hookrightarrow \mathcal{O}^{\mathrm{r}}_{\tilde{\pi}^{-1}\mathrm{U}}.$$

By assumption, $\tilde{\beta}'_{\{t_0\}\times\tilde{\pi}^{-1}U}$ extends to a monomorphism $\beta : \mathrm{V} \hookrightarrow \mathrm{W}$, namely the same β from which α was derived. Hence, possibly after restricting T further to a neighbourhood of t_0, $\tilde{\beta}'$ extends to a monomorphism $\tilde{\beta} : \tilde{\mathrm{V}}|_{\tilde{\pi}^{-1}\mathrm{U}} \hookrightarrow \tilde{\mathrm{W}}$. The maps thus obtained

$$\mathrm{Def}_{\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})}(\mathrm{W} \twoheadrightarrow \mathrm{Q})(\mathrm{T}) \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathrm{Def}(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$$

are obviously functorial and inverse to each other. Hence, we have proved statement *(i)*.

By [20, Theorem 4.1], $\mathrm{Def}(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$ is formally smooth if the hyper-ext group $\mathbb{E}\mathrm{xt}^2(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$ vanishes. The kernel of α is just $\mathrm{V}(-\mathrm{N}_\xi)$. We let M be the cokernel and obtain a short exact sequence of complexes

$$0 \to \mathrm{K}^\bullet_0 \to \mathrm{K}^\bullet_1 \to \mathrm{K}^\bullet_2 \to 0$$

with

$$\mathrm{K}^0_0 = 0 \longrightarrow \mathrm{K}^1_0 = 0 \longrightarrow \mathrm{K}^2_0 = \mathrm{M};$$

$$\mathrm{K}^0_1 = \mathrm{V} \xrightarrow{\alpha} \mathrm{K}^1_1 = \mathrm{W}|_{\mathrm{N}_\xi} \longrightarrow \mathrm{K}^2_1 = \mathrm{M};$$

$$\mathrm{K}_2^0 = \mathrm{V} \xrightarrow{\alpha} \mathrm{K}_2^1 = \mathrm{W}|_{\mathrm{N}_\xi} \longrightarrow 0.$$

We infer an exact sequence

$$\mathbb{E}\mathrm{xt}^2(\mathrm{V}, \mathrm{K}_1^\bullet) \to \mathbb{E}\mathrm{xt}^2(\mathrm{V}, \mathrm{K}_2^\bullet) \to \mathbb{E}\mathrm{xt}^3(\mathrm{V}, \mathrm{K}_0^\bullet).$$

We recall that if $\mathrm{K}^\bullet \to \mathrm{I}^\bullet$ is a quasi-isomorphism from a complex $\mathrm{K}^\bullet$ to a complex of injective modules, then $\mathbb{E}\mathrm{xt}^\mathrm{i}(\mathrm{V}, \mathrm{K}^\bullet)$ is equal to $\mathrm{H}^\mathrm{i}(\mathrm{Hom}(\mathrm{V}, -)(\mathrm{I}^\bullet))$. As a first consequence, $\mathbb{E}\mathrm{xt}^3(\mathrm{V}, \mathrm{K}_0^\bullet) = \mathrm{Ext}^1(\mathrm{V}, \mathrm{M})$. Since $\mathrm{V}|_{\mathrm{N}_\xi} \to \mathrm{W}|_{\mathrm{N}_\xi}$ is generically an isomorphism due to the definition of $\mathcal{V}'_\Delta(\mathrm{m}, \mathrm{r})$, we have M as a torsion module on N_ξ. Since V is torsion free, we have a short exact sequence

$$0 \to \mathrm{V} \longrightarrow \mathrm{V}^{\vee\vee} \longrightarrow \mathrm{T} \to 0,$$

where the bidual $\mathrm{V}^{\vee\vee}$ is locally free as a reflexive sheaf on a smooth surface and where T is a torsion sheaf concentrated in a subscheme of S of codimension 2, the algebraic set of singularities of V (cf. [35, II.1.1]). Therefore, we may choose the fibre N_ξ of κ in such a way that V is locally free in a neighbourhood of N_ξ and thus locally free restricted to N_ξ. We have this freedom of choice because of statement (i). In the exact sequence

$$0 \to \mathcal{T}or_1^{\mathcal{O}_\mathrm{S}}(\mathrm{V}, \mathcal{O}_{\mathrm{N}_\xi}) \longrightarrow \mathrm{V}(-\mathrm{N}_\xi) \longrightarrow \mathrm{V} \longrightarrow \mathrm{V}|_{\mathrm{N}_\xi} \to 0$$

we see that $\mathcal{T}or_1^{\mathcal{O}_\mathrm{S}}(\mathrm{V}, \mathcal{O}_{\mathrm{N}_\xi})$ is a torsion sheaf with support of codimension 2 and $\mathrm{V}(-\mathrm{N}_\xi)$ is torsion free; therefore $\mathcal{T}or_1^{\mathcal{O}_\mathrm{S}}(\mathrm{V}, \mathcal{O}_{\mathrm{N}_\xi})$ vanishes. Hence, every locally free resolution $\mathcal{L}_\bullet \to \mathrm{V}$ restricts to a locally free resolution $\mathcal{L}_\bullet|_{\mathrm{N}_\xi} \to \mathrm{V}|_{\mathrm{N}_\xi}$, and with [13, Proposition III.6.5 and 6.9] we obtain

$$\begin{aligned}&\mathrm{Ext}^1_\mathrm{S}(\mathrm{V}, \mathrm{M}) = \mathrm{H}^0(\mathrm{S}, \mathcal{E}xt^1(\mathrm{V}, \mathrm{M})) = \mathrm{H}^0(\mathrm{S}, \mathrm{H}^\mathrm{i}(\mathcal{L}_\bullet^\vee \otimes \mathrm{M})) = \mathrm{H}^0(\mathrm{N}_\xi, \mathrm{H}^\mathrm{i}(\mathcal{L}_\bullet^\vee|_{\mathrm{N}_\xi} \otimes \mathrm{M}))\\ &= \mathrm{H}^0(\mathrm{N}_\xi, \mathcal{E}xt^1(\mathrm{V}|_{\mathrm{N}_\xi}, \mathrm{M})) = \mathrm{Ext}^1_{\mathrm{N}_\xi}(\mathrm{V}|_{\mathrm{N}_\xi}, \mathrm{M}),\end{aligned}$$

where we have used that $\mathrm{M} = \mathrm{M}(\mathrm{H})$ for any divisor H. We obtain that $\mathrm{Ext}^1(\mathrm{V}, \mathrm{M})$ is equal to $\mathrm{H}^1(\mathrm{N}_\xi, (\mathrm{V}|_{\mathrm{N}_\xi})^\vee \otimes \mathrm{M})$, which vanishes, as $(\mathrm{V}|_{\mathrm{N}_\xi})^\vee \otimes \mathrm{M}$ is concentrated in finitely many points.

Thus, we have $\mathbb{E}\mathrm{xt}^2(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$ as a quotient of $\mathbb{E}\mathrm{xt}^2(\mathrm{V}, \mathrm{K}_1^\bullet)$. Since any complex of injective modules quasi-isomorphic to $\mathrm{K}_1^\bullet$ is in fact an injective resolution of $\mathrm{V}(-\mathrm{N}_\xi)$, we have $\mathbb{E}\mathrm{xt}^2(\mathrm{V}, \mathrm{K}_1^\bullet) = \mathrm{Ext}^2(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi))$. Therefore it remains to show that $\mathrm{Ext}^2(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi))$ vanishes.

We keep T as the cokernel of $\mathrm{V} \hookrightarrow \mathrm{V}^{\vee\vee}$, which is either zero or a torsion sheaf concentrated in finitely many points. We consider the spectral sequence

$$\mathrm{E}_2^{\mathrm{p},\mathrm{q}} = \mathrm{H}^{\mathrm{q}}(\mathrm{S}, \mathcal{E}xt^{\mathrm{p}}(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi))) \Rightarrow \mathrm{Ext}^{\mathrm{p}+\mathrm{q}}(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)).$$

Since V is torsion free, it is of projective dimension 1 and $\mathcal{E}xt^2(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi))$ vanishes; in particular $\mathrm{H}^0(\mathrm{S}, \mathcal{E}xt^2(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi))) = 0$. Because $\mathrm{V} \hookrightarrow \mathrm{V}^{\vee\vee}$ is an isomorphism outside the support of T, $\mathcal{E}xt^1(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi))$ is concentrated in finitely many points. Hence, $\mathrm{H}^1(\mathrm{S}, \mathcal{E}xt^1(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)))$ vanishes, too. Finally, we take a look at $\mathrm{H}^2(\mathrm{S}, \mathcal{H}om(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)))$: From the short exact sequence

$$0 \to \mathrm{V} \to \mathrm{V}^{\vee\vee} \to \mathrm{T} \to 0,$$

we obtain the short exact sequence

$$0 \to \mathcal{H}om(\mathrm{V}^{\vee\vee}, \mathrm{V}(-\mathrm{N}_\xi)) \to \mathcal{H}om(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)) \to \mathcal{E}xt^1(\mathrm{T}, \mathrm{V}(-\mathrm{N}_\xi)) \to 0$$

and infer an exact sequence

$$\begin{aligned} \mathrm{H}^2(\mathrm{S}, \mathcal{H}om(\mathrm{V}^{\vee\vee}, \mathrm{V}(-\mathrm{N}_\xi))) &\to \mathrm{H}^2(\mathrm{S}, \mathcal{H}om(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi))) \\ &\to \mathrm{H}^2(\mathrm{S}, \mathcal{E}xt^1(\mathrm{T}, \mathrm{V}(-\mathrm{N}_\xi))). \end{aligned}$$

Because of Serre duality and $\mathrm{V}^{\vee\vee}$ being locally free, we have

$$\begin{aligned} \mathrm{H}^2(\mathrm{S}, \mathcal{H}om(\mathrm{V}^{\vee\vee}, \mathrm{V}(-\mathrm{N}_\xi))) &= \mathrm{H}^2(\mathrm{S}, \mathrm{V}^\vee \otimes \mathrm{V}(-\mathrm{N}_\xi)) \\ &= \mathrm{H}^0(\mathrm{S}, \mathrm{V}^{\vee\vee} \otimes \mathrm{V}^\vee(-2,-1)(\mathrm{E}'))^\vee. \end{aligned}$$

Since $\mathrm{V}^{\vee\vee} \otimes \mathrm{V}^\vee(-2,0)(\mathrm{E}')$ is torsion free and trivial along general fibres of π, the module $\pi_*\mathrm{V}^{\vee\vee} \otimes \mathrm{V}^\vee(-2,-1)(\mathrm{E}')$ vanishes, as does the cohomology module $\mathrm{H}^0(\mathrm{S}, \mathrm{V}^{\vee\vee} \otimes \mathrm{V}^\vee(-2,-1)(\mathrm{E}'))$. But $\mathrm{H}^2(\mathrm{S}, \mathcal{E}xt^1(\mathrm{T}, \mathrm{V}(-\mathrm{N}_\xi)))$ is also zero, since the sheaf $\mathcal{E}xt^1(\mathrm{T}, \mathrm{V}(-\mathrm{N}_\xi))$ is concentrated on the support of T, too. Thus, all cohomology groups $\mathrm{H}^{\mathrm{q}}(\mathrm{S}, \mathcal{E}xt^{2-\mathrm{q}}(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)))$ vanish, and we obtain $\mathrm{Ext}^2(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)) = 0$, and hence the formal smoothness of $\mathrm{Def}(\mathrm{V}, \alpha : \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi})$. $\square$

4.3 Natural description of the completion

Of course, the described completion of the moduli space $\mathcal{V}(\mathrm{m},\mathrm{r})$ is natural. But the question we discuss here is, which objects are precisely parametrised by $\mathcal{V}_\Delta(\mathrm{m},\mathrm{r})$ or, equivalently, which are the objects in the complement

$$\mathcal{V}_\Delta(\mathrm{m},\mathrm{r}) - \mathcal{V}(\mathrm{m},\mathrm{r})?$$

We recall the definition of $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})$ from the proof of Theorem 4.5 as the subfunctor of $\mathrm{Quot}^{\mathrm{f}}(\mathcal{O}^{\mathrm{r}}(\mathrm{m},0))$ corresponding to all those points

$$0 \to \mathrm{V} \longrightarrow \mathcal{O}^{\mathrm{r}}(\mathrm{m},0) \longrightarrow \mathrm{Q} \to 0,$$

where Q is concentrated in finitely many fibres of π.

Theorem 4.7

$$\mathcal{V}_\Delta(\mathrm{m},\mathrm{r}) = \mathcal{V}'_\Delta(\mathrm{m},\mathrm{r}).$$

Proof. We put $\mathrm{W} = \mathcal{O}^{\mathrm{r}}(\mathrm{m},0)$ and consider a point

$$(\beta) = (0 \to \mathrm{V} \xrightarrow{\beta} \mathrm{W} \longrightarrow \mathrm{Q} \to 0)$$

in $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})$. We will say that this point is *deformable* to another point

$$(\beta_1) = (0 \to \mathrm{V}_1 \xrightarrow{\beta} \mathrm{W} \longrightarrow \mathrm{Q}_1 \to 0)$$

if there is a connected double-based algebraic space $(\mathrm{T},\mathrm{t}_0,\mathrm{t}_1)$ and an element

$$(\tilde\beta) = (0 \to \tilde{\mathrm{V}} \xrightarrow{\beta} \tilde{\mathrm{W}} \longrightarrow \tilde{\mathrm{Q}} \to 0)$$

in $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})(\mathrm{T})$ such that $(\tilde\beta)|_{\{\mathrm{t}_0\}\times\mathrm{S}} \cong (\beta)$ and $(\tilde\beta)|_{\{\mathrm{t}_1\}\times\mathrm{S}} \cong (\beta_1)$. Because of Lemma 4.6, we know that $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})$ is smooth. Hence, all its connected components are irreducible, and it is enough to show that (β) is deformable to a point in $\mathcal{V}(\mathrm{m},\mathrm{r})$. This is a consequence of the following three lemmas. □

Lemma 4.8 *A point*

$$(\beta) = (0 \to \mathrm{V} \xrightarrow{\beta} \mathrm{W} \longrightarrow \mathrm{Q} \to 0)$$

in $\mathcal{V}'_\Delta(\mathrm{m},\mathrm{r})$ *belongs to* $\mathcal{V}(\mathrm{m},\mathrm{r})$ *iff* V *is locally free and free along the fibre* N_∞ *of* κ *and the fibre* F_∞ *of* π.

Proof. As we have seen before, β induces a framing of V in a point on N_∞ and hence a framing $\alpha : \mathrm{V}|_{\mathrm{N}_\infty} \cong \mathcal{O}^{\mathrm{r}}_{\mathrm{N}_\infty}$. The element $(\mathrm{V}, \alpha) \in \mathcal{V}(\mathrm{m}, \mathrm{r})(\operatorname{Spec} \mathrm{k})$ is obviously mapped to (β) under $\mathcal{V}(\mathrm{m}, \mathrm{r}) \hookrightarrow \mathrm{Quot}^{\mathrm{f}}\mathrm{W}$. $\square$

Lemma 4.9 *Any point*

$$(\beta) = (0 \to \mathrm{V} \xrightarrow{\beta} \mathrm{W} \longrightarrow \mathrm{Q} \to 0)$$

in $\mathcal{V}'_\Delta(\mathrm{m}, \mathrm{r})$ *is deformable to a point*

$$(\beta_1) = (0 \to \mathrm{V}_1 \xrightarrow{\beta} \mathrm{W} \longrightarrow \mathrm{Q}_1 \to 0)$$

such that V_1 *is free along* N_∞ *and* F_∞*.*

Proof. Evidently, it is enough to show that (β) deforms to a (β_1), where V_1 is free along the generic fibre of κ, as such a point is easily deformed to a point which is trivial along two given special fibres of κ and π.

We choose a fibre N_ξ of κ such that V is locally free in a neighbourhood of N_ξ. During the proof of Lemma 4.6.*(ii)*, we have shown the vanishing of $\mathcal{T}or_1^{\mathcal{O}_\mathrm{S}}(\mathrm{V}, \mathcal{O}_\mathrm{L})$. Therefore, we have the exact sequence of complexes

$$0 \to (\mathrm{V}(-\mathrm{N}_\xi) \to 0) \longrightarrow (\mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi}) \longrightarrow (\mathrm{V}|_{\mathrm{N}_\xi} \to \mathrm{W}|_{\mathrm{N}_\xi}) \to 0$$

and obtain an exact sequence

$$\mathbb{E}\mathrm{xt}^1(\mathrm{V}, \mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi}) \longrightarrow \mathbb{E}\mathrm{xt}^1(\mathrm{V}, \mathrm{V}|_{\mathrm{N}_\xi} \to \mathrm{W}|_{\mathrm{N}_\xi}) \longrightarrow \mathrm{Ext}^2(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)).$$

As we have seen in the proof of Lemma 4.6.*(ii)*, the vanishing of $\mathcal{T}or_1^{\mathcal{O}_\mathrm{S}}(\mathrm{V}, \mathcal{O}_\mathrm{L})$ implies $\mathbb{E}\mathrm{xt}^1(\mathrm{V}, \mathrm{V}|_{\mathrm{N}_\xi} \to \mathrm{W}|_{\mathrm{N}_\xi}) = \mathbb{E}\mathrm{xt}^1(\mathrm{V}|_{\mathrm{N}_\xi}, \mathrm{V}|_{\mathrm{N}_\xi} \to \mathrm{W}|_{\mathrm{N}_\xi})$, and it was also shown there that $\mathrm{Ext}^2(\mathrm{V}, \mathrm{V}(-\mathrm{N}_\xi)) = 0$. The module $\mathbb{E}\mathrm{xt}^1(\mathrm{V}|_{\mathrm{N}_\xi}, \mathrm{V}|_{\mathrm{N}_\xi} \to \mathrm{W}|_{\mathrm{N}_\xi})$ equals $\mathrm{Ext}^1(\mathrm{V}|_{\mathrm{N}_\xi}, \mathrm{W}|_{\mathrm{N}_\xi})/\mathrm{V}|_{\mathrm{N}_\xi}))$, which is naturally identified with the space of infinitesimal deformations of the quotient $\mathrm{V}|_{\mathrm{N}_\xi} \hookrightarrow \mathrm{W}|_{\mathrm{N}_\xi}$. Hence, any infinitesimal deformation of $\mathrm{V}|_{\mathrm{N}_\xi} \hookrightarrow \mathrm{W}|_{\mathrm{N}_\xi}$ lifts to an infinitesimal deformation of the framed module $\mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi}$. Because of the formal smoothness shown in Lemma 4.6.*(ii)*, this yields that any deformation of $\mathrm{V}|_{\mathrm{N}_\xi} \hookrightarrow \mathrm{W}|_{\mathrm{N}_\xi}$ lifts to a deformation of the framed module sheaf $\mathrm{V} \to \mathrm{W}|_{\mathrm{N}_\xi}$. And by Lemma 4.6.*(i)*, we have that any deformation of $\mathrm{V}|_{\mathrm{N}_\xi} \hookrightarrow \mathrm{W}|_{\mathrm{N}_\xi}$ lifts to a deformation of the quotient $\mathrm{V} \hookrightarrow \mathrm{W}$. As $\mathrm{V}|_{\mathrm{N}_\xi} \hookrightarrow \mathrm{W}|_{\mathrm{N}_\xi}$ is deformable to $\mathcal{O}^{\mathrm{r}}_{\mathrm{N}_\xi} \hookrightarrow \mathrm{W}|_{\mathrm{N}_\xi}$, we have shown our claim. □

Lemma 4.10 *Any point*

$$(\beta) = (0 \to \mathrm{V} \xrightarrow{\beta} \mathrm{W} \longrightarrow \mathrm{Q} \to 0)$$

in $\mathcal{V}'_\Delta(\mathrm{m}, \mathrm{r})$ *is deformable to a point*

$$(\beta_1) = (0 \to \mathrm{V}_1 \xrightarrow{\beta_1} \mathrm{W} \longrightarrow \mathrm{Q}_1 \to 0)$$

such that V_1 *is locally free.*

Proof. For any torsion free sheaf $\mathcal{F}$ we define $\mathrm{T}(\mathcal{F})$ as the quotient of $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee}$, which is a torsion sheaf concentrated in finitely many points. We will proceed now in several steps.

Step 1. V *is deformable to a torsion free sheaf* V_1*, such that* $\mathrm{T}(\mathrm{V}_1)$ *is isomorphic to a direct sum* $\mathrm{k}(\mathrm{P}_1) \oplus \ldots \oplus \mathrm{k}(\mathrm{P}_\mathrm{q})$ *with* $\mathrm{P}_\mathrm{i} \neq \mathrm{P}_\mathrm{j}$ *for* $\mathrm{i} \neq \mathrm{j}$.

This is true since $\mathrm{Quot}^\mathrm{q}(\mathrm{V}^{\vee\vee})$ is irreducible by [22, Theorem 6.A.1], and since the morphism $\mathrm{Quot}^\mathrm{q}(\mathrm{V}^{\vee\vee}) \to \mathrm{Sym}^\mathrm{q}\mathrm{S}$ is surjective.

Step 2. Let V *be a torsion free sheaf of rank* $\mathrm{r} \geq 2$ *with* $\mathrm{T}(\mathrm{V}) = \mathrm{k}(\mathrm{P}_1) \oplus \ldots \oplus \mathrm{k}(\mathrm{P}_\mathrm{q})$, *and let* $\sigma : \mathrm{S}' \to \mathrm{S}$ *be the blowing up in* $\{\mathrm{P}_1, \ldots, \mathrm{P}_\mathrm{q}\}$*. There is a locally free sheaf* V' *on* S' *such that* $\sigma_*\mathrm{V}' = \mathrm{V}$ *and* $\mathrm{R}^1\sigma_*\mathrm{V}' = 0$.

As this problem is local around singular points of V, we may assume $\mathrm{q} = 1$. We fix an open neighbourhood U of $\mathrm{P} = \mathrm{P}_1$, where $\mathrm{V}^{\vee\vee}$ is trivial. Hence, we may choose an isomorphism $\varphi : \mathrm{V}|_\mathrm{U} \cong \mathcal{J}_{\mathrm{P}|\mathrm{U}} \oplus \mathcal{O}^{\mathrm{r}-1}_\mathrm{U}$. We obtain $(\sigma|_{\tilde{\mathrm{U}}})^*\mathrm{V}|_\mathrm{U} \cong \mathcal{O}_{\tilde{\mathrm{U}}}(-\mathrm{E}) \oplus \mathcal{O}^{\mathrm{r}-1}_{\tilde{\mathrm{U}}}$, where $\tilde{\mathrm{U}} = \sigma^{-1}\mathrm{U}$ and $\mathrm{E} = \sigma^{-1}\{\mathrm{P}\}$. We embed this into

$$\mathrm{V}'_{\tilde{\mathrm{U}}} = \mathcal{O}_{\tilde{\mathrm{U}}}(-\mathrm{E}) \oplus \mathcal{O}_{\tilde{\mathrm{U}}}(\mathrm{E}) \oplus \mathcal{O}^{\mathrm{r}-2}_{\tilde{\mathrm{U}}}$$

and define V' by gluing together $(\sigma|_{\mathrm{S}'-\mathrm{E}})^*\mathrm{V}$ and $\mathrm{V}'_{\tilde{\mathrm{U}}}$ by $(\sigma|_{\tilde{\mathrm{U}}-\mathrm{E}})^*\varphi|_{\mathrm{U}-\mathrm{P}}$. Clearly, this locally free sheaf V' meets our requirements.

Step 4. For the vector sheaf V' *constructed in Step 3. we have* $\mathrm{c}_\bullet \mathrm{V}' = \sigma^* \mathrm{c}_\bullet \mathrm{V}$.

By construction, the determinant of V' along E is trivial. Hence, $\det \mathrm{V}$ is a pullback under σ. Since determinants are not altered by changes in codimension 2, and since $\sigma : \mathrm{S}' - \mathrm{E} \to \mathrm{S} - \{\mathrm{P}\}$ is an isomorphism, we obtain $\det \mathrm{V}' = \sigma^* \det \mathrm{V}$.

Because of the vanishing of the higher direct images, the Leray spectral sequence associated to V' and σ degenerates and we have $\mathrm{H}^{\mathrm{i}}(\mathrm{S}, \mathrm{V}) = \mathrm{H}^{\mathrm{i}}(\mathrm{S}', \mathrm{V}')$ for $\mathrm{i} = 0, 1, 2$. Therefore, the Euler characteristics coincide, as do the second Chern classes due to Hirzebruch-Riemann-Roch.

Step 5. The vector sheaf V' *constructed in Step 3 is deformable to a vector sheaf* V_1 *on* S' *which is free along* E.

We define $\pi' = \pi \circ \sigma$ and note that V' is free along the generic fibre of π' by construction. Hence, $\mathrm{Ext}^2(\mathrm{V}', \mathrm{V}'(-\mathrm{E})) = \mathrm{H}^0(\mathrm{S}', \mathcal{E}nd(\mathrm{V}')(\mathrm{E} + \omega_{\mathrm{S}'}))^\vee$ vanishes, and the map $\mathrm{Ext}^1(\mathrm{V}', \mathrm{V}') \to \mathrm{Ext}^1(\mathrm{V}'|_{\mathrm{E}}, \mathrm{V}'|_{\mathrm{E}})$ is surjective. This implies, that any infinitesimal deformation of $\mathrm{V}'|_{\mathrm{E}}$ lifts to an infinitesimal deformation of V', and since $\mathrm{Ext}^2(\mathrm{V}', \mathrm{V}')$ also vanishes, this is true for all deformations. Now we conclude with the observation that $\mathcal{O}_{\mathrm{E}}(1) \oplus \mathcal{O}_{\mathrm{E}}(-1) \oplus \mathcal{O}_{\mathrm{E}}^{\mathrm{r}-2}$ is deformable to $\mathcal{O}_{\mathrm{E}}^{\mathrm{r}}$.

Step 6. V *is deformable to a locally free sheaf* V_1 *in a family of sheaves, which are all free along the generic fibre of* π.

Because of the previous steps, we may assume that V occurs as the direct image of a vector sheaf V' on a blowing up $\sigma : \mathrm{S}' \to \mathrm{S}$ with $\mathrm{R}^1\sigma_* \mathrm{V}' = 0$, and that there is a flat family $\tilde{\mathrm{V}}' \in \mathcal{V}ec(\mathrm{T} \times \mathrm{S}')$ with the following properties:

- There is a point $\mathrm{t}_0 \in \mathrm{T}$ with $\tilde{\mathrm{V}}'|_{\{\mathrm{t}_0\} \times \mathrm{S}'} \cong \mathrm{V}'$;
- For all other closed points $\mathrm{t} \in \mathrm{T}$, $\tilde{\mathrm{V}}'|_{\{\mathrm{t}\} \times \mathrm{S}'}$ is free along the exceptional divisor of σ;
- All members of the family are free along the generic fibre of π'.

For $\sigma_{\mathrm{T}} : \mathrm{T} \times \mathrm{S}' \to \mathrm{T} \times \mathrm{S}$, we have that $\tilde{\mathrm{V}} = \sigma_{\mathrm{T}*} \tilde{\mathrm{V}}'$ is flat over T, as the Hilbert polynomials are constant. Because of base change being possible in flat families, this yields in particular the desired deformation of V.

Step 7. (β) is deformable to (β_1) with V_1 locally free.

We consider the flat family $\tilde{\mathrm{V}}$ from Step 6 and define $\tilde{\pi} : \mathrm{T} \times \mathrm{S} \to \mathrm{T} \times \mathrm{N}$. Since the members of the family are trivial along the generic fibre of π, we may consider the determinantal divisor $\tilde{\Delta}$ associated to $\tilde{\mathrm{V}}(-\mathrm{T} \times \mathrm{N}_\infty)$ and $\tilde{\pi}$. If U denotes the complement of the support of $\tilde{\Delta}$, then β induces an isomorphism

$$\tilde{\beta}_{\tilde{\pi}^{-1}\mathrm{U}} : \tilde{\mathrm{V}}|_{\tilde{\pi}^{-1}\mathrm{U}} \cong (\mathcal{O}_\mathrm{T} \boxtimes \mathrm{W})|_{\tilde{\pi}^{-1}\mathrm{U}}.$$

In the same way as in the proof of Lemma 4.6.*(i)*, this extends to a monomorphism $\tilde{\beta} : \tilde{(\mathrm{V})} \hookrightarrow \mathcal{O}_\mathrm{T} \boxtimes \mathrm{W}$ flat over T, which is obviously an element in $\mathcal{V}'_\Delta(\mathrm{m}, \mathrm{r})$ and proves our claim. $\square$

Due to Theorem 4.7, the following forms of degeneration of a point

$$0 \to \mathrm{V} \longrightarrow \mathcal{O}^\mathrm{r}(\mathrm{m}, 0) \longrightarrow \mathrm{Q} \to 0$$

in $\mathcal{V}_\Delta(\mathrm{m}, \mathrm{r}) - \mathcal{V}(\mathrm{m}, \mathrm{r})$ are feasible:

(1) V is not free along N_∞ and F_∞;

(2) V is not even free along the generic fibre of κ;

(3) V is not locally free.

Of course, there are certain restrictions. For instance, if L is a fibre of π or κ, and P is a smooth point of L, then $\dim_\mathrm{k} \mathrm{H}^1(\mathrm{L}, \mathrm{V}|_\mathrm{L}(-\mathrm{P})) \leq \mathrm{m}$. If $\mathrm{V}|_\mathrm{L}$ is locally free, then there are m smooth points $\mathrm{P}_1, \ldots, \mathrm{P}_\mathrm{m}$ on L such that we have vanishings $\mathrm{H}^0(\mathrm{L}, \mathrm{V}|_\mathrm{L}(-\sum \mathrm{P}_\mathrm{i})) = \mathrm{H}^1(\mathrm{L}\mathrm{V}|_\mathrm{L}(\sum \mathrm{P}_\mathrm{i})) = 0$. And the length of $\mathrm{V}^{\vee\vee}/\mathrm{V}$ is always less or equal to m.

On the other side, all the mentioned cases of degenerations actually occur. We consider for example $\mathrm{S} = \mathrm{N} \times \mathrm{F}$. Due to a Serre construction, any element in $\mathcal{V}(1, 2)$ is given by a non-trivial extension

$$0 \to \mathcal{O}_\mathrm{S} \longrightarrow \mathrm{V} \longrightarrow \mathcal{J}_\mathrm{P} \to 0,$$

where $\mathcal{J}_\mathrm{P}$ is the ideal sheaf of a point P which lies in the complement of $\mathrm{N}_\infty \cup \mathrm{F}_\infty$, and a framing $\mathrm{V}|_{\mathrm{N}_\infty} \cong \mathcal{O}^\mathrm{r}_{\mathrm{N}_\infty}$.

If we choose such an extension with $\mathrm{P} \in \mathrm{N}_\infty \cup \mathrm{F}_\infty$, we obtain a degeneration of type (1). If we let such an extension degenerate to a trivial one, we have a degeneration of type (3). A degeneration of type (2) is given if we consider an embedding of $\mathcal{O}(1,0) \oplus \mathcal{J}_\mathrm{P}(-1,0)$ or of $\mathcal{O}(-1,0) \oplus \mathcal{J}_\mathrm{P}(1,0)$ into $\mathcal{O}^2(1,0)$.

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Tabellarischer Lebenslauf

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Liste der Publikationen

- A geometric example of non-trivially mixed Hodge structure. *erscheint in: Journal of Pure and Applied Algebra, 1997*
- On framed instanton bundles and their deformations. *elektronisches Preprint, Duke-Server alg-geom/9611001, 1996*

Berlin, den 13. Oktober 1997

Erklärung

Hiermit versichere ich, daß ich die vorgelegte Dissertation selbständig und ohne unerlaubte Hilfe angefertigt habe.

Ich erkläre, daß ich die Arbeit erstmalig und nur an der Humboldt-Universität zu Berlin eingereicht habe und mich nicht anderwärts um einen Doktorgrad beworben habe. In dem Promotionsfach besitze ich keinen Doktorgrad.

Der Inhalt der dem Verfahren zugrundeliegenden Promotionsordnung ist mir bekannt.

Andreas Matuschke